

A LITTELMANN PATH MODEL FOR CRYSTALS OF GENERALIZED KAC-MOODY ALGEBRAS.

ANTHONY JOSEPH AND POLYXENI LAMPROU

ABSTRACT. A Littelmann path model is constructed for crystals pertaining to a not necessarily symmetrizable Borcherds-Cartan matrix. Here one must overcome several combinatorial problems coming from the imaginary simple roots. The main results are an isomorphism theorem and a character formula of Borcherds-Kac-Weyl type for the crystals. In the symmetrizable case, the isomorphism theorem implies that the crystals constructed by this path model coincide with those of Jeong, Kang, Kashiwara and Shin obtained by taking $q \rightarrow 0$ limit in the quantized enveloping algebra.

Key words : Crystals, Path Model, character formula.

AMS Classification : 17B37.

1. INTRODUCTION

1.1. The original proof of the Weyl character formula given in 1925 by Weyl following the work of Schur for $\mathfrak{gl}(n)$ underwent a number of simplifications with a particularly notable one due to Bernstein, Gelfand and Gelfand [1]. This proof was shown by Kac [8] to extend to integrable modules for Kac-Moody algebras obtained from a symmetrizable Cartan matrix. For affine Lie algebras the corresponding Weyl denominator formula spectacularly recovered and generalized sum-product identities from number theory due to Fermat, Gauss and Jacobi.

More recently Borcherds [2] showed that the Kac-Moody theory extends with equally beautiful results when imaginary simple roots are permitted. In particular the Bernstein-Gelfand-Gelfand method gives a character formula, somewhat more complicated than the Weyl-Kac formula for unitarizable highest weight modules.

1.2. In 1986, Drinfeld and Jimbo independently introduced quantized enveloping algebras involving a parameter q . A little later Kashiwara [11] considered a $q \rightarrow 0$ limit of these algebras and the integrable modules over them. Interpreting q as the temperature, these modules were deemed to “crystallize” into a simpler form. In particular an integrable highest weight module gives rise to a normal highest weight crystal (which can be viewed as a rather special graph). Since much structure is lost in the process the latter are not uniquely defined by their highest weights. However using the tensor structure, one obtains a unique closed (under tensor product) family of normal highest weight crystals. More recently Jeong, Kang and Kashiwara [4] have extended this theory to include simple imaginary roots (as in

Work supported in part by the European Community RTN network “Liegrits”, Grant No. MRTN - CT - 2003 - 505078 and in part by Minerva Foundation, Germany, Grant No. 8466.

Borcherds) but still with the assumption that the Cartan matrix is symmetrizable (which is needed for quantization).

1.3. Shortly after Kashiwara introduced crystals, Littelmann [14, 15] found a purely combinatorial path model for them based on the Cartan matrix which was no longer required to be symmetrizable. He constructed a closed family of normal highest weight crystals and computed their characters. This was based on Lakshmibai-Seshadri paths, themselves described by Bruhat sequences in the Weyl group together with an integrality condition.

1.4. In this paper we extend Littelmann's path model to include imaginary simple roots. This involves a number of combinatorial complications. Instead of the Weyl group we use a monoid with generators defined by both the real and the imaginary simple roots. Here the presence of non-invertible elements ultimately means that the normal highest weight crystals are not strict subcrystals of the full crystal defined by all possible paths. Besides they are normal only with respect to the real simple roots. This makes it more difficult to show that "generalized" Lakshmibai-Seshadri paths describe the required normal highest weight crystals. It becomes correspondingly more difficult to show that this set of crystals is closed with respect to tensor product. However this being achieved we recover in the symmetrizable case, the Kashiwara crystals by uniqueness. Finally we prove a version of Littelmann's combinatorial character formula for the crystals in this family. This does not need the Cartan matrix to be symmetrizable, though in any case a very similar formula to that of Borcherds is obtained.

1.5. Unfortunately Littelmann's combinatorial formula does not recover the Weyl denominator formula (known to hold in the non-symmetrizable totally real case by independent work of Kumar [13] and Mathieu [16]). The question this entails and many others remain open.

Acknowledgements. This work started when the second author was visiting the University of Cologne as a Liegrits predoc. She would like to take the opportunity to thank P. Littelmann for his hospitality and his guidance during her stay.

2. PRELIMINARIES

2.1. **Generalized Kac-Moody algebras.** Unless otherwise specified all numerical values are assumed rational. In particular, all vector spaces are over \mathbb{Q} . We denote by \mathbb{N} the set of natural numbers and we set $\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$.

2.1.1. Let I be a countable index set. We call $A = (a_{ij})_{i,j \in I}$ a *Borcherds-Cartan matrix* if the following are satisfied :

- (1) $a_{ii} = 2$ or $a_{ii} \in -\mathbb{N}^+$ for all i ,
- (2) $a_{ij} \in -\mathbb{N}^+$, for all $i \neq j$,
- (3) $a_{ij} = 0$ if and only if $a_{ji} = 0$.

We call an index i real if $a_{ii} = 2$ and we denote by I^{re} the set of real indices. Otherwise, we call an index i imaginary and we denote by $I^{im} = I \setminus I^{re}$, the set of imaginary indices.

If $I = I^{re}$ and is finite, then A is a generalized Cartan matrix in the language of [7, Section 1.1]. The matrix A is called *symmetrizable* if there exists a diagonal matrix $S = \text{diag}\{s_i \in \mathbb{N}^+ \mid i \in I\}$ such that SA is symmetric.

2.1.2. Let \mathfrak{g} be the generalized Kac-Moody algebra associated to a Borcherds-Cartan matrix A , \mathfrak{h} a fixed Cartan subalgebra of \mathfrak{g} , $\Pi = \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ the set of simple roots, $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\} \subset \mathfrak{h}$ the set of simple coroots such that $\alpha_i^\vee(\alpha_j) = a_{ij}$ and Δ the root system of \mathfrak{g} (for more details see [2, 3]).

2.1.3. Let $P = \{\lambda \in \mathfrak{h}^* \mid \alpha_i^\vee(\lambda) \in \mathbb{Z}, \text{ for all } i \in I\}$ be the weight lattice of \mathfrak{g} , $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ be the root lattice and $Q^+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i$. Of course $\Delta \subset Q \subset P$. Set $P^+ = \{\lambda \in P \mid \alpha_i^\vee(\lambda) \geq 0, \text{ for all } i \in I\}$.

2.1.4. Define a partial order in Q by setting $\beta \succ \gamma$ if and only if $\beta - \gamma \in Q^+$. Let $\Delta^+ = \{\beta \in \Delta \mid \beta \succ 0\}$ be the set of positive roots and $\Delta^- = -\Delta^+$ the set of negative roots. One has that $\Delta = \Delta^+ \sqcup \Delta^-$.

2.1.5. For all $i \in I$ let r_i be the linear map $r_i : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ defined by

$$r_i(x) = x - \alpha_i^\vee(x)\alpha_i.$$

Note that r_i is a reflection (and thus $r_i^2 = \text{id}$) if and only if $i \in I^{re}$. Otherwise, if $i \in I^{im}$, r_i has infinite order. Set $T = \langle r_i \mid i \in I \rangle$ to be the monoid generated by all the r_i , $i \in I$ and denote by id its neutral element. Let W be the group generated by the reflections r_i , $i \in I^{re}$ and call it the Weyl group of \mathfrak{g} . Then of course W lies in T . For any $\tau \in T$ we may write $\tau = r_{i_1} r_{i_2} \cdots r_{i_\ell}$ where $i_j \in I$, for all j with $1 \leq j \leq \ell$. We call this a reduced expression if ℓ takes its minimal value which we define to be the reduced length $\ell(\tau)$ of τ .

2.1.6. For all $i \in I$, we define r_i on \mathfrak{h} by :

$$r_i(h) = h - h(\alpha_i)\alpha_i^\vee.$$

One checks that $(r_i h)(r_i \lambda) = h(\lambda)$ for all $h \in \mathfrak{h}$ and all $\lambda \in \mathfrak{h}^*$.

2.1.7. Set $\mathcal{C} = \{\mu \in \mathfrak{h}^* \mid \alpha_i^\vee(\mu) \geq 0, \text{ for all } i \in I^{re}\}$, the set of dominant elements of \mathfrak{h}^* . (Notice that we consider only real indices). We call a weight in \mathcal{C} a dominant weight. One has that $P^+ \subset \mathcal{C}$. Notice that $-\alpha_i \in P^+ \subset \mathcal{C}$ for all $i \in I^{im}$. By [7, Proposition 3.12], for all $\lambda \in \mathcal{C}$ one has $W\lambda \cap \mathcal{C} = \{\lambda\}$. Choose $\rho \in \mathfrak{h}^*$ such that $\alpha_i^\vee(\rho) = \frac{1}{2}a_{ii}$. Then $\rho \in \mathcal{C}$, but in general $\rho \notin P^+$.

2.1.8. Let $\lambda \in \mathcal{C}$ and denote by W_λ the stabilizer of λ in W . Then W_λ is generated by the simple reflections which stabilize λ , that is $W_\lambda = \langle r_i \mid r_i \lambda = \lambda \rangle$. Even when $|I^{re}| = \infty$, the proof is as in [7, Proposition 3.12].

2.1.9. Denote by $\Pi_{re} = \{\alpha_i \mid i \in I^{re}\}$ and by $\Pi_{im} = \{\alpha_i \mid i \in I^{im}\}$ the sets of real and imaginary simple roots respectively.

Lemma. Take $\alpha_i, \alpha_j \in \Pi$ and $w, \tilde{w} \in W$. If $w\alpha_i = \tilde{w}\alpha_j$, then $w\alpha_i^\vee = \tilde{w}\alpha_j^\vee$.

Proof. If $\alpha_i, \alpha_j \in \Pi_{re}$, the assertion obtains from [7, Section 5.1]. Suppose $\alpha_i \in \Pi_{im}$. Since $w\alpha_i \in \alpha_i + \mathbb{N}\Pi_{re}$, the hypothesis forces $\alpha_i = \alpha_j$. It then suffices to prove the assertion for $\tilde{w} = \text{id}$ and $w \in \text{Stab}_W(\alpha_i)$. Since $-\alpha_i \in \mathcal{C}$, by section 2.1.8, we can write $w = r_{i_1} \cdots r_{i_k}$, with $\alpha_{i_t}^\vee(\alpha_i) = 0$, for all t , with $1 \leq t \leq k$. Then $\alpha_i^\vee(\alpha_{i_t}) = 0$, for all t , with $1 \leq t \leq k$, so $w \in \text{Stab}_W(\alpha_i^\vee)$, as required. \square

Definition. By the above lemma, we may define $\beta^\vee \in \mathfrak{h}$, for all $\beta \in W\Pi$, through $\beta^\vee = w\alpha_i^\vee$, given $\beta = w\alpha_i$.

2.1.10. Take $i \in I^{re}$. Through [7, Lemma 3.8] we obtain $r_i(\Delta^+ \setminus \{\alpha_i\}) \subset \Delta^+ \setminus \{\alpha_i\}$. In particular, Δ is W -stable. Call a root $\beta \in W\Pi$ real if $\beta^\vee(\beta) = 2$ and imaginary if $\beta^\vee(\beta) \leq 0$. Set $\Delta_{re} = W\Pi_{re}$ and $\Delta_{im} = W\Pi_{im}$. Define also $\Delta_{re}^+ = \Delta_{re} \cap \Delta^+$, $\Delta_{re}^- = -\Delta_{re}^+$ and notice that $\Delta_{im} \subset \Delta^+$. In general $\Delta_{re} \sqcup (\Delta_{im} \sqcup -\Delta_{im}) \subset \Delta$ is a strict inclusion. However, its complement in Δ does not play any role in our analysis.

2.1.11. One could roughly say that everything we know about the Weyl group and the real roots in the Kac-Moody case, also holds for the generalized Kac-Moody algebras. The imaginary roots need some attention. The following result will be repeatedly used in the sequel.

Lemma. Take $i \in I^{im}$, then

- (1) $\beta^\vee(\alpha_i) \leq 0$, for all $\beta \in \Delta_{re}^+ \sqcup \Delta_{im}$,
- (2) $\alpha_i^\vee(\beta) \leq 0$, for all $\beta \in Q^+$.

Proof. Indeed, for (1) take $\beta = w\alpha_j$; then $\beta^\vee(\alpha_i) = \alpha_j^\vee(w^{-1}\alpha_i)$ and $w^{-1}\alpha_i \in \alpha_i + \mathbb{N}\Pi_{re}$, because $-\alpha_i \in \mathcal{C}$. Hence the assertion for $\alpha_j \in \Pi_{im}$. For $\alpha_j \in \Pi_{re}$, one must show that $(w\alpha_j)^\vee \in \mathbb{N}\Pi_{re}^\vee$. This is stated in [7, Section 5.1]. Finally, (2) is an immediate consequence of the properties of the matrix A . \square

2.2. Dominant elements in $T\lambda$. In this section we give a characterization of the dominant weights in the T -orbit $T\lambda$ of a weight $\lambda \in P^+$.

2.2.1. **Lemma.** For all $\lambda \in P^+$ one has that $T\lambda \subset \lambda - Q^+$. In particular, $\alpha_i^\vee(\mu) \geq 0$ for all $\mu \in T\lambda$ and all $i \in I^{im}$.

Proof. We will prove by induction on $\ell(\tau)$ that

$$\tau\lambda \in W\lambda - \mathbb{N}\Delta_{im}.$$

Then since $W\lambda \subset \lambda - \mathbb{N}\Delta_{re}^+$, as noted in section 2.1.8, and $\Delta_{im} \subset \Delta^+$ by section 2.1.10, we will have that $T\lambda \subset \lambda - \mathbb{N}\Delta^+ = \lambda - Q^+$.

For $\tau = \text{id}$ the statement is obvious. Let $\tau\lambda = w\lambda - \beta = \lambda - \gamma$, with $\beta \in \mathbb{N}\Delta_{im}$ and $\gamma \in \mathbb{N}\Delta^+$. Take $i \in I^{im}$, then since by lemma 2.1.11 (2), $\alpha_i^\vee(\lambda - \gamma) \geq 0$ one has

$$r_i\tau\lambda \in \tau\lambda - \mathbb{N}\alpha_i \subset W\lambda - \mathbb{N}\Delta_{im}.$$

Take $i \in I^{re}$. By section 2.1.10 we have that $r_i\Delta_{im} \subset \Delta_{im}$ and so

$$r_i\tau\lambda \in r_iw\lambda - \mathbb{N}\Delta_{im} \subset W\lambda - \mathbb{N}\Delta_{im}.$$

Hence the assertion. \square

2.2.2. Lemma. *The stabilizer of $\lambda \in P^+$ in T is generated by the r_i , $i \in I$ which stabilize λ , that is $\text{Stab}_T(\lambda) = \langle r_i \mid \alpha_i^\vee(\lambda) = 0 \rangle$.*

Proof. Set $S := \langle r_i \mid \alpha_i(\lambda) = 0 \rangle$. Clearly, $S \subset \text{Stab}_T(\lambda)$. Let $\tau \in \text{Stab}_T(\lambda)$; we will show that $\tau \in S$. We argue by induction on $\ell(\tau)$. If $\tau = r_i$, for $i \in I$, the assertion is clear. Let $\tau \in \text{Stab}_T(\lambda)$ be such that $\ell(\tau) > 1$ and write $\tau = r_i\tau'$, with $\ell(\tau') < \ell(\tau)$. Then, by the previous lemma $r_i\tau'\lambda = r_i(w\lambda - \beta) = r_i(\lambda - \gamma)$, with $\beta \in \mathbb{N}\Delta_{im}$ and $\gamma \in \mathbb{N}\Delta^+$.

If $i \in I^{im}$, $\alpha_i^\vee(\tau'\lambda) \geq 0$, which forces $\tau'\lambda = \lambda$ and $\alpha_i^\vee(\tau'\lambda) = 0$. In particular, $\alpha_i^\vee(\lambda) = 0$ and $\tau' \in \text{Stab}_T(\lambda)$. Then $\tau' \in S$, by the induction hypothesis and $r_i \in S$, hence $\tau \in S$.

If $i \in I^{re}$, $\lambda = r_i\tau'\lambda = r_iw\lambda - r_i\beta$, hence $\beta = 0$ and $r_iw \in W_\lambda \subset S$. Then $\tau'\lambda = w\lambda = \lambda$, so by the induction hypothesis $\tau' \in S$ and since $r_i \in S$, we get $\tau \in S$. Hence the assertion. \square

2.2.3. Let $\lambda \in P^+$ and recall section 2.1.5. One would like to know which elements in $T\lambda$ are dominant. Here we remark that by lemma 2.2.1 one has that $T\lambda \cap P^+ = T\lambda \cap \mathcal{C}$. By section 2.1.7, for all $w \in W$, with $w \notin W_\lambda$, $w\lambda$ is not dominant. On the other hand, notice that for all dominant $\mu \in T\lambda$ and all $i \in I^{im}$, $r_i\mu$ is also dominant. Indeed, for all $j \in I$ we have that $\alpha_j^\vee(r_i\mu) = \alpha_j^\vee(\mu) - \alpha_i^\vee(\mu)a_{ji} \geq 0$, since μ is dominant and $a_{ji} \leq 0$. In particular, $r_{i_1}r_{i_2} \cdots r_{i_k}\lambda$ is dominant for all $i_1, i_2, \dots, i_k \in I^{im}$.

Lemma. *Let $\mu \in T\lambda \cap P^+$ and $i \in I^{im}$ and assume that $r_iw\mu \notin P^+$ for some $w \neq \text{id}$ in W . Then $r_iw\mu = r_jr_iw'\mu$, for some $j \in I^{re}$ with $w' := r_jw$ and $\ell(w') = \ell(w) - 1$. Consequently there exist $w_1, w_2 \in W$ such that $w = w_1w_2$ and $\ell(w) = \ell(w_1) + \ell(w_2)$, with $r_iw_1 = w_1r_i$. Moreover, $\mu' := r_iw_2\mu$ is dominant and $r_iw\mu = w_1\mu'$.*

Proof. Since $\mu \in T\lambda$, one has that $r_i\tau\mu \in T\lambda$ and so, by lemma 2.2.1, $\alpha_j^\vee(r_i\tau\mu) \geq 0$ for all $j \in I^{im}$. Now by assumption $r_iw\mu$ is not dominant, hence there exists a $j \in I^{re}$ such that $\alpha_j^\vee(r_iw\mu) < 0$. This gives

$$\alpha_j^\vee(w\mu) - \alpha_i^\vee(w\mu)a_{ji} < 0,$$

hence

$$(1) \quad \alpha_j^\vee(w\mu) < \alpha_i^\vee(w\mu)a_{ji}.$$

Now $r_jw\mu \in T\lambda$ and so by lemma 2.2.1 $\alpha_i^\vee(r_jw\mu) \geq 0$ which in turn gives :

$$(2) \quad \alpha_i^\vee(w\mu) - \alpha_j^\vee(w\mu)a_{ij} \geq 0,$$

Suppose that a_{ji} (and so a_{ij}) is not equal to zero and hence $a_{ij}, a_{ji} < 0$. Then equations (1) and (2) give

$$\alpha_i^\vee(w\mu)(1 - a_{ij}a_{ji}) > 0.$$

But this is impossible since $1 - a_{ij}a_{ji} \leq 0$ and again by lemma 2.2.1, $\alpha_i^\vee(w\mu) \geq 0$. We conclude that $a_{ij} = a_{ji} = 0$, which implies that r_i and r_j commute and $\alpha_j^\vee(r_i w\mu) = \alpha_j^\vee(w\mu) < 0$. Since $\mu \in P^+$, the last inequality forces $w = r_j w'$, for some $w' \in W$ with $\ell(w') = \ell(w) - 1$. Finally, $r_i w = r_i r_j w' = r_j r_i w'$. By repeating the procedure for $r_i w'\mu$, the last assertion follows. \square

2.2.4. Lemma. *Let $\mu \in P^+$ and $\tau \in T$. If $\alpha_j^\vee(\tau\mu) < 0$, for some $j \in I^{re}$, then $\ell(r_j\tau) < \ell(\tau)$.*

Proof. Let

$$(3) \quad \tau = w_0 r_{i_1} w_1 \cdots w_{k-1} r_{i_k} w_k,$$

with $w_t \in W$, $0 \leq t \leq k$ and $i_s \in I^{im}$, $1 \leq s \leq k$ be a reduced expression of τ . By the previous lemma, we can write $r_{i_k} w_k \mu$ as $w'_k \mu'$, with $\mu' = r_{i_k} w''_k \mu \in P^+$, $r_{i_k} w'_k = w'_k r_{i_k}$ and

$$(4) \quad \ell(w_k) = \ell(w'_k) + \ell(w''_k).$$

Thus we get a new expression for τ :

$$\tau = w_0 r_{i_1} w_1 \cdots r_{i_{k-1}} w'_{k-1} r_{i_k} w''_k,$$

where $w'_{k-1} = w_{k-1} w'_k$. By (4) and since the expression (3) of τ is reduced we get $\ell(w'_{k-1}) = \ell(w_{k-1}) + \ell(w'_k)$. Repeating this procedure, we obtain $\tau\mu = w'_0 \nu$ and $\nu = \tau'\mu \in P^+$, with $\ell(\tau) = \ell(w'_0) + \ell(\tau')$. Let $j \in I^{re}$. Then $\alpha_j^\vee(\tau\mu) < 0$ implies that $\alpha_j^\vee(w'_0 \nu) < 0$ and so $\ell(r_j w'_0) < \ell(w'_0)$ which in turn gives that $\ell(r_j \tau) = \ell(r_j w'_0 \tau') < \ell(w'_0 \tau') = \ell(\tau)$. \square

2.2.5. For any $\mu \in T\lambda$, $\lambda \in P^+$, call $\tau\lambda$ a reduced expression of μ if $\mu = \tau\lambda$ and for every τ' such that $\mu = \tau'\lambda$ one has that $\ell(\tau) \leq \ell(\tau')$. We have the following result :

Corollary. *An element $\mu \neq \lambda$ in $T\lambda$ is dominant if and only if every reduced expression of μ is of the form $r_i \tau \lambda$, for $\tau \in T$, $i \in I^{im}$.*

Proof. Suppose that $\mu = \tau\lambda$ is a reduced expression of μ . As in the proof of lemma 2.2.4 one can write $\mu = w_0 \nu$, with $\nu = r_i \tau' \lambda \in P^+$, $i \in I^{im}$, $\tau' \in T$ and $\tau = w_0 r_i \tau'$, where lengths add. If μ is dominant, then $w_0 \in W_\nu$. But then $\mu = w_0 r_i \tau' \lambda = r_i \tau' \lambda$ which implies that $w_0 = \text{id}$, hence every reduced expression of μ starts with some r_i with $i \in I^{im}$. If μ is not dominant, then by lemma 2.2.3, there exists a reduced expression of μ starting with r_j , where $j \in I^{re}$. Hence the assertion. \square

2.2.6. We may express this consequence of lemma 2.2.4 in the following fashion. Choose $\mu \in P^+$ and $\tau \in T$ written as in equation (3). Call τ a dominant reduced expression if τ is reduced and successively the $\ell(w_k), \ell(w_{k-1}), \dots, \ell(w_0)$ take their minimal values. Set $\tau' = r_{i_1} w_1 \cdots r_{i_k} w_k$. Then $\tau\mu$ is dominant if and only if $w_0 \in \text{Stab}_W(\tau'\mu)$.

3. GENERALIZED CRYSTALS

3.1. The notion of a crystal.

3.1.1. **Definition.** A generalized crystal B is a set endowed with the maps $\text{wt} : B \rightarrow P$, $\varepsilon_i, \varphi_i : B \rightarrow \mathbb{Z} \cup \{-\infty\}$, $e_i, f_i : B \rightarrow B \cup \{0\}$ satisfying the rules :

- (1) For all $i \in I$ and all $b \in B$, $\varphi_i(b) = \varepsilon_i(b) + \alpha_i^\vee(\text{wt } b)$.
- (2) For all $i \in I$ if $b, e_i b \in B$, then $\text{wt}(e_i b) = \text{wt } b + \alpha_i$.
- (3) For all $i \in I$ if $b, e_i b \in B$, then $\varepsilon_i(e_i b) = \varepsilon_i(b) - 1$ for $i \in I^{re}$ and $\varepsilon_i(e_i b) = \varepsilon_i(b)$ if $i \in I^{im}$.
- (4) For all $i \in I$ and all $b, b' \in B$ one has $b' = e_i b$ if and only if $f_i b' = b$.
- (5) If for $b \in B, i \in I, \varphi_i(b) = -\infty$, then $e_i b = f_i b = 0$.
- (6) For all $i \in I^{im}$ and all $b \in B$, $\varepsilon_i(b) \in (-\mathbb{N}) \sqcup \{-\infty\}$ and $\varphi_i(b) \in \mathbb{N} \sqcup \{-\infty\}$.

3.1.2. Remarks.

- (1) The axioms imply the following further properties. First $\varphi_i(e_i b) = \varphi_i(b) + 1$, if $i \in I^{re}$ and $\varphi_i(e_i b) = \varphi_i(b) + a_{ii}$, if $i \in I^{im}$. Second (a) $\text{wt } f_i b = \text{wt } b - \alpha_i$, (b) $\varepsilon_i(f_i b) = \varepsilon_i(b) + 1$, if $i \in I^{re}$ and $\varepsilon_i(f_i b) = \varepsilon_i(b)$, if $i \in I^{im}$, (c) $\varphi_i(f_i b) = \varphi_i(b) - 1$ if $i \in I^{re}$ and $\varphi_i(f_i b) = \varphi_i(b) - a_{ii}$ if $i \in I^{im}$.
- (2) The crystal graph of a crystal B is the graph having vertices the elements of B and arrows $b \xrightarrow{i} b'$ if $f_i b = b'$.
- (3) This definition is due to Jeong, Kang, Kashiwara and Shin [4, 5]. We omit the term “generalized” in the sequel.

3.1.3. For any $\mu \in P$ set $B_\mu = \{b \in B \mid \text{wt } b = \mu\}$. If all B_μ are finite, define the formal character of B to be

$$\text{char } B := \sum_{b \in B} e^{\text{wt } b} = \sum_{\mu \in P} |B_\mu| e^\mu$$

Call a crystal B *upper normal* if $\varepsilon_i(b) = \max\{n \in \mathbb{N} \mid e_i^n b \neq 0\}$ for all $i \in I^{re}$, *lower normal* if $\varphi_i(b) = \max\{n \in \mathbb{N} \mid f_i^n b \neq 0\}$ for all $i \in I^{re}$ and *normal* if it is both upper and lower normal.

Denote by \mathcal{F} the monoid generated by the $f_i; i \in I$. A crystal B is called a *highest weight crystal* of highest weight λ if there exists an element $b_\lambda \in B$, such that $\text{wt } b_\lambda = \lambda$ and $B = \mathcal{F}b$. Notice that this implies that $e_i b = 0$ for all $i \in I$, but the converse can fail. Despite the obvious analogy to highest weight modules, this condition is rather weak (see also remark in section 3.2.1). Indeed, given a crystal B and an element $b \in B_\lambda$, we obtain a highest weight subcrystal $\mathcal{F}b$ of B , simply by declaring $e_i b' = 0$, whenever $e_i b' \notin \mathcal{F}b$.

3.1.4. Let \mathcal{B} be the set of crystals B which for all $b \in B$ and all $i \in I^{im}$ satisfy :

- (1) $\alpha_i^\vee(\text{wt } b) \geq 0$,
- (2) $\varepsilon_i(b) = 0$ and consequently $\varphi_i(b) = \alpha_i^\vee(\text{wt } b)$,
- (3) $f_i b \neq 0$ if and only if $\varphi_i(b) > 0$.

3.1.5. Lemma. *Let $B \in \mathcal{B}$ and take $i \in I^{im}$, $b \in B$. If $\alpha_i^\vee(\text{wt } b) \leq -a_{ii}$, then $e_i b = 0$. In particular, $e_i b = 0$ if $\alpha_i^\vee(\text{wt } b) = 0$.*

Proof. Suppose that $e_i b \neq 0$, then $f_i(e_i b) \neq 0$ by 3.1.1 (4), and so $0 < \varphi_i(e_i b) = \alpha_i^\vee(\text{wt } e_i b)$. By 3.1.1 (2), $\text{wt } e_i b = \text{wt } b + \alpha_i$ and so $\alpha_i^\vee(\text{wt } e_i b) > 0$ implies that $\alpha_i^\vee(\text{wt } b) > -a_{ii}$. \square

Remark. The converse of the above lemma is false.

3.1.6. Definition. A morphism ψ of crystals B_1, B_2 is a map

$$\psi : B_1 \longrightarrow B_2 \cup \{0\}$$

such that:

- (1) $\text{wt}(\psi(b)) = \text{wt } b$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, $\varphi_i(\psi(b)) = \varphi_i(b)$ for all $i \in I$.
- (2) $\psi(e_i b) = e_i \psi(b)$, if $e_i b \neq 0$.
- (3) $\psi(f_i b) = f_i(\psi(b))$, if $f_i b \neq 0$.

One says that B_1 is a subcrystal of B_2 if ψ is an embedding. An embedding is said to be *strict*, if e_i commutes with ψ for all $i \in I$. If ψ is a strict embedding, then B_1 is said to be a *strict subcrystal* of B_2 . The crystal graph of a subcrystal B_1 of B_2 is obtained by removing the arrows between vertices of B_1 and vertices of $B_2 \setminus B_1$ in the crystal graph of B_2 .

3.2. Crystal tensor product.

3.2.1. Definition. Let B_1, B_2 be two crystals. Their tensor product $B_1 \otimes B_2$ is $B_1 \times B_2$ as a set, with crystal operations defined as follows. Set $b = b_1 \otimes b_2$ with $b_1 \in B_1$, $b_2 \in B_2$. Then :

- (1) $\text{wt } b = \text{wt } b_1 + \text{wt } b_2$.
- (2) $\varepsilon_i(b) = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \alpha_i^\vee(\text{wt } b_1)\}$.
- (3) $\varphi_i(b) = \max\{\varphi_i(b_1) + \alpha_i^\vee(\text{wt } b_2), \varphi_i(b_2)\}$.
- (4) For all $i \in I$,

$$(a) \quad f_i b = \begin{cases} f_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes f_i b_2, & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

- (5) For all $i \in I^{re}$,

$$(b) \quad e_i b = \begin{cases} e_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes e_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

and for all $i \in I^{im}$ we set

$$(c) \quad e_i b = \begin{cases} e_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) - a_{ii}, \\ 0, & \text{if } \varepsilon_i(b_2) < \varphi_i(b_1) \leq \varepsilon_i(b_2) - a_{ii}, \\ b_1 \otimes e_i b_2, & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

It is straightforward to verify that $B_1 \otimes B_2$ endowed with the above operations is indeed a crystal [5, Lemma 2.10]. Moreover, as in the Kac-Moody case, the tensor product of two normal crystals is a normal crystal.

Remark. If $\mathcal{F}b_\lambda$ and $\mathcal{F}b_\mu$ are highest weight crystals, it is not obvious that $\mathcal{F}(b_\lambda \otimes b_\mu)$ is a strict subcrystal of $\mathcal{F}b_\lambda \otimes \mathcal{F}b_\mu$.

3.2.2. Let B_1, B_2 be crystals in \mathcal{B} , form their tensor product $B := B_1 \otimes B_2$ and let $b := b_1 \otimes b_2 \in B$. Take $i \in I^{im}$. The formulae 3.2.1 (a) and (c) simplify as follows :

$$(a') \quad f_i b = \begin{cases} f_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > 0, \\ b_1 \otimes f_i b_2, & \text{if } \varphi_i(b_1) = 0, \end{cases}$$

and

$$(c') \quad e_i b = \begin{cases} e_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > 0, \\ b_1 \otimes e_i b_2, & \text{if } \varphi_i(b_1) = 0. \end{cases}$$

Indeed, equation (a') above immediately obtains from 3.2.1 (a) since $\varphi_i(b_1) \geq 0 = \varepsilon_i(b_2)$. For $e_i b$ notice that the only case where equation 3.2.1 (c) and equation (c') above can differ is when $0 < \varphi_i(b_1) \leq -a_{ii}$. But then by lemma 3.1.5 one has that $e_i b_1 = 0$ and so $e_i(b_1 \otimes b_2) = 0$ by either (c) or (c').

The set \mathcal{B} is closed under tensor products. Indeed notice that $\alpha_i^\vee(\text{wt } b) = \alpha_i^\vee(\text{wt } b_1) + \alpha_i^\vee(\text{wt } b_2) \geq 0$ and $\varepsilon_i(b) = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \alpha_i^\vee(\text{wt } b_1)\} = 0$. Now if $\varphi_i(b) > 0$, then either $\varphi_i(b_1) > 0$ and $f_i b = (f_i b_1) \otimes b_2 \neq 0$ or $\varphi_i(b_1) = 0, \varphi_i(b_2) > 0$ and $f_i b = b_1 \otimes f_i b_2 \neq 0$. On the other hand, if $\varphi_i(b) = 0$, then $\varphi_i(b_1) = \varphi_i(b_2) = 0$ which implies that $f_i b_1 = f_i b_2 = 0$ and so $f_i b = 0$. We conclude that $f_i b \neq 0$ if and only if $\varphi_i(b) > 0$ as required.

3.3. The crystal $B(\infty)$.

3.3.1. For any index $i \in I$ we define the *elementary crystal* B_i [5, Example 2.14] to be the set $B_i = \{b_i(-n) \mid n \in \mathbb{N}\}$ with crystal operations:

$$\begin{aligned} \text{wt } b_i(-n) &= -n\alpha_i, \\ e_i b_i(-n) &= b_i(-n+1), \quad f_i b_i(-n) = b_i(-n-1) \\ e_j b_i(-n) &= f_j b_i(-n) = 0, \quad \text{if } i \neq j \\ \varepsilon_i(b_i(-n)) &= n, \quad \varphi_i(b_i(-n)) = -n, \quad \text{if } i \in I^{re} \\ \varepsilon_i(b_i(-n)) &= 0, \quad \varphi_i(b_i(-n)) = -na_{ii}, \quad \text{if } i \in I^{im} \\ \varepsilon_j(b_i(-n)) &= \varphi_j(b_i(-n)) = -\infty, \quad \text{if } i \neq j, \end{aligned}$$

where we have set $b_i(-n) = 0$ for all $n < 0$.

3.3.2. **Theorem.** *There exists a unique (up to isomorphism) crystal, denoted by $B(\infty)$, with the properties :*

- (1) *There exists an element $b_0 \in B(\infty)$ of weight zero.*
- (2) *The set of weights of $B(\infty)$ lies in $-Q^+$.*
- (3) *For any element $b \in B(\infty)$ with $b \neq b_0$, there exists some $i \in I$ such that $e_i b \neq 0$.*
- (4) *For all $i \in I$ there exists a unique strict embedding $\Psi_i : B(\infty) \longrightarrow B(\infty) \otimes B_i$, sending b_0 to $b_0 \otimes b_i(0)$.*

For A symmetrizable and $a_{ii} \in -2\mathbb{N}^+$ if $i \in I^{im}$, the above result is due to Jeong, Kang and Kashiwara [5, Theorem 4.1]. Their proof is not combinatorial. We shall prove it combinatorially and in general by constructing a path model. The description of $B(\infty)$ which results is given in section 3.3.4 below.

3.3.3. Let $J = \{i_1, i_2, \dots\}$ where $i_j \in I$ is a countable sequence with the property that for all $i \in I$ and all $j \in \mathbb{N}^+$, there exists $k > j$ such that $i_k = i$. It is convenient to assume that $i_j \neq i_{j+1}$ for all $j \in \mathbb{N}^+$. Set $B(k) = B_{i_k} \otimes \dots \otimes B_{i_1}$ and for $k \leq l$, let $\psi_{k,l} : B(k) \rightarrow B(l)$ be the map $b \mapsto b_{i_k}(0) \otimes \dots \otimes b_{i_{l+1}}(0) \otimes b$. Let $B_J(\infty)$ be the inductive limit of the family $\{B(k)\}_{k \geq 1}$. Then $B_J(\infty)$ is the crystal in which an element b takes the form

$$b = \dots \otimes b_{i_2}(-m_2) \otimes b_{i_1}(-m_1),$$

with $m_k \in \mathbb{N}$ and $m_k = 0$ for $k \gg 0$. The crystal structure of $B_J(\infty)$ is given explicitly in [5, Example 2.17]. This is described in section 9.3, where some further properties of $B_J(\infty)$ are discussed.

3.3.4. Let B be a crystal satisfying properties (1)-(4) of theorem 3.3.2. Then b_0 is the unique element of weight zero in B . Indeed, if $b \neq b_0$ and $\text{wt } b = 0$ then $e_i b \neq 0$ for some $i \in I$. But then $\text{wt } e_i b = \alpha_i \notin -Q^+$ contradicting property (2). It follows that $B = \mathcal{F}b_0$.

Iterating (4) we have a strict embedding :

$$B \hookrightarrow B \otimes B_{i_1} \hookrightarrow B \otimes B_{i_2} \otimes B_{i_1} \hookrightarrow \dots \hookrightarrow B \otimes B_{i_r} \otimes \dots \otimes B_{i_2} \otimes B_{i_1},$$

for all $r > 0$. There exists $N > 0$ such that any element $b \in B$ takes the form

$$b_0 \otimes b_{i_N}(-m_N) \otimes \dots \otimes b_{i_1}(-m_1).$$

Associating $\dots \otimes b_{i_{N+1}}(0) \otimes b_{i_N}(-m_N) \otimes \dots \otimes b_{i_1}(-m_1)$ to b we obtain a strict embedding $B \hookrightarrow B_J(\infty)$. Now $B_J(\infty)$ admits a unique element b_∞ of weight zero given by taking all the $m_k = 0$ for all $k \in \mathbb{N}^+$. Then B is the strict subcrystal of $B_J(\infty)$ generated by b_∞ . We conclude that a crystal satisfying (1)-(4) of theorem 3.3.2 is unique.

4. A PATH MODEL FOR CRYSTALS DEFINED BY A BORCHERDS-CARTAN MATRIX

According to our general conventions, all intervals are considered in \mathbb{Q} , that is we write $[a, b]$ for $\{c \in \mathbb{Q} \mid a \leq c \leq b\}$. Let X be a topological space. A function $\pi : [0, 1] \rightarrow X$ is said to be continuous (or just a path) if it is the restriction of a continuous function on the real interval. Actually, we shall mainly use piecewise linear functions.

Let \mathbb{P} be the set of paths $\pi : [0, 1] \rightarrow \mathbb{Q}P$ such that $\pi(0) = 0$ and $\pi(1) \in P$. We consider two paths $\pi, \pi' \in \mathbb{P}$ equivalent if $\pi = \pi'$ up to parametrization, i.e. if there exists a non-decreasing continuous function $\phi : [0, 1] \rightarrow [0, 1]$ such that $\pi(\phi(t)) = \pi'(t)$ for all $t \in [0, 1]$. We call $\pi(1)$ the weight of the path $\pi(t)$ and sometimes we write $\text{wt } \pi = \pi(1)$.

4.1. The operators f_i, e_i .

4.1.1. For all $\pi \in \mathbb{P}$ and all $i \in I$, set $h_i^\pi(t) := \alpha_i^\vee(\pi(t))$, $t \in [0, 1]$ and let m_i^π be the minimal integral value of the function h_i^π , that is

$$m_i^\pi = \min\{h_i^\pi(t) \cap \mathbb{Z} | t \in [0, 1]\}.$$

(Notice that since $\pi(0) = 0$, one has that $h_i^\pi(0) = 0$, hence the function h_i^π attains integral values.) The action of f_i, e_i for $i \in I$ is defined in the following sections.

4.1.2. Let $f_+^i(\pi) \in [0, 1]$ be maximal such that $h_i^\pi(f_+^i(\pi)) = m_i^\pi$. Suppose $f_+^i(\pi) < 1$. Since $\pi(1) \in P$ and so $h_i^\pi(1) \in \mathbb{Z}$, it follows that there exists $f_-^i(\pi) \in [f_+^i(\pi), 1]$ minimal such that $h_i^\pi(f_-^i(\pi)) = m_i^\pi + 1$. Then $(f_i\pi)(t)$ is defined to be the path :

$$(f_i\pi)(t) = \begin{cases} \pi(t), & t \in [0, f_+^i(\pi)], \\ \pi(f_+^i(\pi)) + r_i(\pi(t) - \pi(f_+^i(\pi))), & t \in [f_+^i(\pi), f_-^i(\pi)], \\ \pi(t) - \alpha_i, & t \in [f_-^i(\pi), 1]. \end{cases}$$

Otherwise (if $f_+^i(\pi) = 1$), we set $f_i\pi = 0$.

4.1.3. Take $i \in I^{re}$, and let $e_+^i(\pi) \in [0, 1]$ be minimal such that $h_i^\pi(e_+^i(\pi)) = m_i^\pi$. If $e_+^i(\pi) > 0$ let $e_-^i(\pi) \in [0, e_+^i(\pi)]$ be maximal such that $h_i^\pi(e_-^i(\pi)) = m_i^\pi + 1$. The path $e_i\pi$ is then defined by :

$$(e_i\pi)(t) = \begin{cases} \pi(t), & t \in [0, e_-^i(\pi)], \\ \pi(e_-^i(\pi)) + r_i(\pi(t) - \pi(e_-^i(\pi))), & t \in [e_-^i(\pi), e_+^i(\pi)], \\ \pi(t) + \alpha_i, & t \in [e_+^i(\pi), 1]. \end{cases}$$

Otherwise (if $e_+^i(\pi) = 0$), we set $e_i\pi = 0$.

4.1.4. Take $i \in I^{im}$. Define $r_i^{-1} : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ to be the map :

$$r_i^{-1}(x) = x + \frac{1}{1 - a_{ii}} \alpha_i^\vee(x) \alpha_i.$$

One checks that $r_i r_i^{-1} = r_i^{-1} r_i = \text{id}$. Recall the number $f_+^i(\pi)$ defined in section 4.1.2 and set $e_-^i(\pi) := f_+^i(\pi)$. If $e_-^i(\pi) = 1$ or $h_i^\pi(t) < m_i^\pi + 1 - a_{ii}$ for all $t \in [e_-^i(\pi), 1]$ or $h_i^\pi(t) \leq m_i^\pi - a_{ii}$ for some $t \in [e_+^i(\pi), 1]$, set $e_i\pi = 0$. Otherwise, let $e_+^i(\pi) \in [e_-^i(\pi), 1]$ be minimal such that $h_i^\pi(e_+^i(\pi)) = m_i^\pi + 1 - a_{ii}$ and set :

$$(e_i\pi)(t) = \begin{cases} \pi(t), & t \in [0, e_-^i(\pi)], \\ \pi(e_-^i(\pi)) + r_i^{-1}(\pi(t) - \pi(e_-^i(\pi))), & t \in [e_-^i(\pi), e_+^i(\pi)], \\ \pi(t) + \alpha_i, & t \in [e_+^i(\pi), 1]. \end{cases}$$

4.1.5. Remarks.

- (1) The definition of f_i, e_i for $i \in I^{re}$ is as in [14, Section 2]. Notice that in [14] the condition under which Littelmann sets $f_i\pi = 0$ is that $h_i^\pi(1) - h_i^\pi(f_+^i(\pi)) < 1$. This is equivalent to equality in $h_i^\pi(1) \geq m_i^\pi$ and so to $f_+^i(\pi) = 1$ if we consider only paths with endpoint in P .
- (2) It is easy to verify that for $e_i, f_i, i \in I$ defined above, $f_i\pi = \pi'$ if and only if $e_i\pi' = \pi$. For $i \in I^{re}$, this is done in [15].
- (3) If $f_i\pi \neq 0$, one has that $\text{wt } f_i\pi = \text{wt } \pi - \alpha_i$. Similarly, if $e_i\pi \neq 0$ then $\text{wt } e_i\pi = \text{wt } \pi + \alpha_i$.

4.1.6. **Lemma.** *Take $i \in I$ and let $\pi \in \mathbb{P}$ be such that $f_i\pi \neq 0$.*

- (1) *If $i \in I^{im}$, then $m_i^{f_i\pi} = m_i^\pi$ and $f_+^i(f_i\pi) = f_+^i(\pi)$, whereas $f_-^i(f_i\pi) \leq f_-^i(\pi)$, with equality if and only if $a_{ii} = 0$. In particular, $f_-^k\pi \neq 0$, for all $k \geq 0$.*
- (2) *If $i \in I^{re}$, then $m_i^{f_i\pi} = m_i^\pi - 1$ and $f_+^i(f_i\pi) = f_-^i(\pi)$. In particular, since $h_i^{f_i\pi}(1) = h_i^\pi(1) - 2$, there exists $k \in \mathbb{N}$ such that $f_-^k\pi = 0$.*

Proof. Consider (1) and let $i \in I^{im}$. By definition, $h_i^\pi(t) \cap \mathbb{Z} \geq h_i^\pi(f_+^i(\pi)) = m_i^\pi$ and this inequality is strict for $t > f_+^i(\pi)$. We will compute the function $h_i^{f_i\pi}(t)$. Recall definition 4.1.2. One has that for $t \in [0, f_+^i(\pi)]$

$$(5) \quad h_i^{f_i\pi}(t) = h_i^\pi(t).$$

Now, for $t \in [f_+^i(\pi), f_-^i(\pi)]$,

$$(6) \quad h_i^{f_i\pi}(t) = h_i^\pi(t) - a_{ii}(h_i^\pi(t) - h_i^\pi(f_+^i(\pi))) \geq h_i^\pi(t).$$

Finally, for $t \in [f_-^i(\pi), 1]$,

$$(7) \quad h_i^{f_i\pi}(t) = h_i^\pi(t) - a_{ii} \geq h_i^\pi(t).$$

By equations (5), (6), (7), we conclude that $h_i^{f_i\pi}(t) \geq h_i^\pi(t)$ for all $t \in [0, 1]$ and so $h_i^{f_i\pi}(t) \cap \mathbb{Z} \geq m_i^\pi$. Since also $h_i^{f_i\pi}(f_+^i(\pi)) = h_i^\pi(f_+^i(\pi)) = m_i^\pi$, we conclude that $m_i^{f_i\pi} = m_i^\pi$. Also $h_i^{f_i\pi}(t) \cap \mathbb{Z} > m_i^\pi$ for $t > f_+^i(\pi)$ and thus $f_+^i(f_i\pi) = f_+^i(\pi)$. By (6) we obtain that $f_-^i(f_i\pi) \leq f_-^i(\pi)$ with equality if and only if $a_{ii} = 0$. Finally, since $f_i\pi = 0$ if and only if $f_+^i(\pi) = 1$ and $f_+^i(f_i\pi) = f_+^i(\pi)$, one has that $f_i\pi \neq 0$ implies that $f_-^2\pi \neq 0$ and inductively, $f_-^k\pi \neq 0$ for all $k \geq 0$. Hence (1).

Statement (2) which we have included for comparison is implicit in [14, Proposition 1.5]. It may be similarly verified by substituting $a_{ii} = 2$ in the first parts of equations (6) and (7). \square

4.2. The Crystal structure of \mathbb{P} .

4.2.1. Assume that for all $\pi \in \mathbb{P}$ and all $i \in I^{im}$ one has that $\alpha_i^\vee(\pi(1)) \geq 0$. We will endow \mathbb{P} with a normal crystal structure. We define the operators $f_i, e_i, i \in I$ as in sections 4.1.2, 4.1.3, 4.1.4. We set $\text{wt } \pi = \pi(1)$. For $i \in I^{re}$ we set $\varepsilon_i(\pi) = -m_i^\pi$. For $i \in I^{im}$, we set $\varepsilon_i(\pi) = 0$. Then φ_i can be recovered by the formula $\varphi_i(\pi) = \varepsilon_i(\pi) + \alpha_i^\vee(\text{wt } \pi)$. From section 4.1.5 and lemma 4.1.6 one checks the following:

Lemma. *The set of paths \mathbb{P} together with the maps $e_i, f_i, \varepsilon_i, \varphi_i, \text{wt}$ for all $i \in I$ defined above, is a normal crystal.*

4.2.2. *Concatenation of paths.* We define the tensor product of $\pi_1, \pi_2 \in \mathbb{P}$ to be the concatenation of the two paths :

$$(\pi_1 \otimes \pi_2)(t) = \begin{cases} \pi_1(t/s), & t \in [0, s], \\ \pi_1(1) + \pi_2(\frac{t-s}{1-s}), & t \in [s, 1], \end{cases}$$

for any rational number $s \in [0, 1]$.

Lemma. *The crystal operations on $\mathbb{P} \otimes \mathbb{P} \subset \mathbb{P}$ satisfy the crystal tensor product rules defined in section 3.2.1.*

Proof. This is straightforward; a point to remark is that $(\pi_1 \otimes \pi_2)(s) \in P$, otherwise the insertion of e_i or f_i will simultaneously change both π_1 and π_2 . \square

5. GENERALIZED LAKSHMIBAI-SESHADRI PATHS

5.1. **Distance of two weights in $T\lambda$.** Notation is as in sections 2.1.1-2.1.10.

5.1.1. Let $\lambda \in P^+$ and let $\mu, \nu \in T\lambda$ be two weights in the T orbit of λ . We write $\mu > \nu$ if there exists a sequence of weights $\mu := \lambda_0, \lambda_1, \dots, \lambda_{s-1}, \lambda_s := \nu$ and positive roots $\beta_1, \dots, \beta_s \in W\Pi \cap \Delta^+ = \Delta_{re}^+ \sqcup \Delta_{im}^+$ such that $\lambda_{i-1} = r_{\beta_i} \lambda_i$ and $\beta_i^\vee(\lambda_i) > 0$, for all i , with $1 \leq i \leq s$. Note that $\mu = r_\beta \nu$, with $\beta \in W\Pi \cap \Delta^+$, one has $\mu > \nu$ if and only if $\beta^\vee(\nu) > 0$.

We call the *distance* of μ and ν and write $\text{dist}(\mu, \nu)$ the maximal length of such sequences. If $\mu = r_\beta \nu > \nu$ and $\text{dist}(\mu, \nu) = 1$ we write $\nu \xleftarrow{\beta} \mu$. Since β is uniquely determined by the pair (μ, ν) , we can omit it and write $\nu \leftarrow \mu$.

5.1.2. **Remarks.**

- (1) If $\beta_i \in \Delta_{re}^+$ for all $i, 1 \leq i \leq s$ then $\text{id} \leftarrow r_{\beta_s} \leftarrow \dots \leftarrow r_{\beta_1} r_{\beta_2} \dots r_{\beta_s}$ is a Bruhat sequence in W/W_λ , where recall that W_λ stands for the stabilizer of λ in W .
- (2) Let $\mu = r_i \nu$ and $\alpha_i^\vee(\nu) > 0$. Then $\text{dist}(\mu, \nu) = 1$. Indeed, note that

$$\nu := \lambda_s \xleftarrow{\beta_s} \lambda_{s-1} \dots \xleftarrow{\beta_2} \lambda_1 \xleftarrow{\beta_1} \lambda_0 =: \mu$$

with $\beta_i \in \Delta^+$ means that $\nu = \mu + \sum_{t=1}^s n_t \beta_t$, with $n_t \in \mathbb{N}^+$. Note that $\mu > \nu$ implies that $\mu \prec \nu$. The converse fails.

5.1.3. The following is exactly as in [15, Lemma 4.1].

Lemma. *Let $\alpha_i \in \Pi_{re}$ be a simple real root and let $\mu \geq \nu$ be two weights in $T\lambda$ with $\lambda \in P^+$. Then :*

- (1) *If $\alpha_i^\vee(\mu) < 0$ and $\alpha_i^\vee(\nu) \geq 0$, then $r_i\mu \geq \nu$ and $\text{dist}(r_i\mu, \nu) < \text{dist}(\mu, \nu)$.*
- (2) *If $\alpha_i^\vee(\mu) \leq 0$ and $\alpha_i^\vee(\nu) > 0$, then $\mu \geq r_i\nu$ and $\text{dist}(\mu, r_i\nu) < \text{dist}(\mu, \nu)$.*
- (3) *If $\alpha_i^\vee(\mu)\alpha_i^\vee(\nu) > 0$, then $r_i\mu \geq r_i\nu$ and $\text{dist}(r_i\mu, r_i\nu) = \text{dist}(\mu, \nu)$.*

5.1.4. **Lemma.** *Let $\mu \geq \nu \in T\lambda$ be such that $\nu := \nu_s \xleftarrow{\beta_s} \nu_{s-1} \xleftarrow{\beta_{s-1}} \dots \xleftarrow{\beta_2} \nu_1 \xleftarrow{\beta_1} \nu_0 =: \mu$ where $\beta_j \in W\Pi \cap \Delta^+$ with $1 \leq j \leq s$ and let $i \in I^{im}$. Then $\alpha_i^\vee(\mu) \geq \alpha_i^\vee(\nu)$ and $\alpha_i^\vee(\mu) = \alpha_i^\vee(\nu)$ if and only if r_i commutes with r_{β_j} for all j with $1 \leq j \leq s$.*

Proof. Since $\mu = \nu - \sum_{j=1}^s \beta_j^\vee(\nu_j)\beta_j \in \nu - \sum_{j=1}^s \mathbb{N}^+\beta_j$ and $\alpha_i^\vee(\beta_j) \leq 0$ for all j , with $1 \leq j \leq s$, we conclude that $\alpha_i^\vee(\mu) \geq \alpha_i^\vee(\nu)$ and equality holds if and only if $\alpha_i^\vee(\beta_j) = 0$ for all j , with $1 \leq j \leq s$. The latter is equivalent to $r_i r_{\beta_j} = r_{\beta_j} r_i$ for all j , with $1 \leq j \leq s$. \square

5.1.5. **Lemma.** *Let $\alpha_i \in \Pi_{im}$ be a simple imaginary root and let $\mu \geq \nu$ be two weights in $T\lambda$ with $\lambda \in P^+$. If $\alpha_i^\vee(\mu) = \alpha_i^\vee(\nu) \geq 0$, then $r_i\mu \geq r_i\nu$ and $\text{dist}(r_i\mu, r_i\nu) = \text{dist}(\mu, \nu)$.*

Proof. Set $\text{dist}(\mu, \nu) = s \geq 1$, then

$$\nu := \nu_s \xleftarrow{\beta_s} \nu_{s-1} \xleftarrow{\beta_{s-1}} \dots \xleftarrow{\beta_2} \nu_1 \xleftarrow{\beta_1} \nu_0 =: \mu,$$

for some $\beta_j \in W\Pi \cap \Delta^+$, where $1 \leq j \leq s$. Since by assumption $\alpha_i^\vee(\mu) = \alpha_i^\vee(\nu)$, lemma 5.1.4 gives that r_i commutes with r_{β_j} for all j , with $1 \leq j \leq s$. Notice that this means that $r_i\mu = r_{\beta_1} \dots r_{\beta_s} r_i\nu$ and $\beta_j^\vee(r_{\beta_{j+1}} \dots r_{\beta_s} r_i\nu) = \beta_j^\vee(r_{\beta_{j+1}} \dots r_{\beta_s} \nu) > 0$. Hence $r_i\mu \geq r_i\nu$ and $\text{dist}(r_i\mu, r_i\nu) \geq s$. Suppose that $\text{dist}(r_i\mu, r_i\nu) > s$. This means that there exist positive roots γ_j , with $1 \leq j \leq t$ and $t > s$, such that

$$r_i\nu := \nu'_t \xleftarrow{\gamma_t} \nu'_{t-1} \xleftarrow{\gamma_{t-1}} \dots \xleftarrow{\gamma_2} \nu'_1 \xleftarrow{\gamma_1} \nu'_0 =: r_i\mu.$$

But our hypothesis $\alpha_i^\vee(\mu) = \alpha_i^\vee(\nu)$ also implies that $\alpha_i^\vee(r_i\mu) = \alpha_i^\vee(r_i\nu)$. By lemma 5.1.4, r_i commutes with r_{γ_j} for all j , with $1 \leq j \leq t$. This gives us $r_i\mu = r_i r_{\gamma_1} \dots r_{\gamma_t} \nu$ and so $\mu = r_{\gamma_1} \dots r_{\gamma_t} \nu$, therefore $\text{dist}(\mu, \nu) \geq t > s$, which is a contradiction. \square

5.2. Generalized Lakshmibai-Seshadri paths.

5.2.1. Let a with $0 < a \leq 1$ be a rational number and let $\mu > \nu$ be two weights in $T\lambda$. An a -chain for the pair (μ, ν) is a sequence of weights in $T\lambda$:

$$\nu := \nu_s \xleftarrow{\beta_s} \nu_{s-1} \xleftarrow{\beta_{s-1}} \dots \xleftarrow{\beta_2} \nu_1 \xleftarrow{\beta_1} \nu_0 =: \mu,$$

such that for all i with $1 \leq i \leq s$:

- (a) $a\beta_i^\vee(\nu_i) \in \mathbb{N}^+$, if $\beta_i \in \Delta_{re}^+$.
- (b) $a\beta_i^\vee(\nu_i) = 1$, if $\beta_i \in \Delta_{im}$.

Observe that if $a = 1$, then condition (a) is automatically satisfied.
For the above a -chain one has

$$(8) \quad a(\mu - \nu) = \sum_{i=0}^{s-1} a(\nu_i - \nu_{i+1}) = - \sum_{i=0}^{s-1} a\beta_i^\vee(\nu_i)\beta_i \in -Q^+.$$

5.2.2. Suppose we have :

- (1) $\boldsymbol{\lambda} = (\lambda_1 > \lambda_2 > \cdots > \lambda_s)$, a sequence of elements in $T\lambda$,
- (2) $\mathbf{a} = (a_0 = 0 < a_1 < \cdots < a_s = 1)$, a sequence of rational numbers,

and set $\pi := (\boldsymbol{\lambda}, \mathbf{a})$ to be the path :

$$(9) \quad \pi(t) = \sum_{i=1}^{j-1} (a_i - a_{i-1})\lambda_i + (t - a_{j-1})\lambda_j, \quad a_{j-1} \leq t \leq a_j.$$

A Generalized Lakshmibai-Seshadri path $\pi = (\boldsymbol{\lambda}, \mathbf{a})$ of shape λ is the path given in (9) such that :

- (a) there exists an a_i -chain for $(\lambda_i, \lambda_{i+1})$ for all i with $1 \leq i \leq s$,
- (b) if $\lambda_s \neq \lambda$ there exists a 1-chain for (λ_s, λ) .

We sometimes write

$$\pi = (\boldsymbol{\lambda}, \mathbf{a}) = (\lambda_1, \lambda_2, \dots, \lambda_s; a_0 = 0, a_1, \dots, a_{s-1}, a_s = 1).$$

For short, we write GLS path for Generalized Lakshmibai-Seshadri path.

5.2.3. Remarks.

- (1) Equation (9) of π can be also written as follows. Let $t \in [a_{j-1}, a_j]$, then :

$$(10) \quad \pi(t) = \sum_{i=1}^{j-1} (a_i - a_{i-1})\lambda_i + (t - a_{j-1})\lambda_j = \sum_{i=1}^{j-1} a_i(\lambda_i - \lambda_{i+1}) + t\lambda_j.$$

- (2) By equation (10) we have that

$$\text{wt } \pi = \pi(1) = \sum_{i=1}^s (a_i - a_{i-1})\lambda_i = \sum_{i=1}^{s-1} a_i(\lambda_i - \lambda_{i+1}) + \lambda_s.$$

Now by equation (8), we have $a_i(\lambda_i - \lambda_{i+1}) \in -Q^+$ for all i with $1 \leq i \leq s-1$ and $\lambda_s \in \lambda - Q^+$. In particular, $\text{wt } \pi$ is an integral weight in the intersection of $\lambda - Q^+$ and the convex hull of $T\lambda$.

5.2.4. *Example.* Let $A = (-k)$ with $k \geq 0$ be an 1×1 matrix, \mathfrak{g} the associated generalized Kac-Moody algebra, α the unique simple (imaginary) root, $r := r_\alpha$. Let λ be a dominant weight in the weight lattice of \mathfrak{g} such that $\alpha^\vee(\lambda) = m > 0$. One checks that the only GLS paths of shape λ are :

$$\begin{aligned} \pi_0 &= (\lambda; 0, 1), \\ \pi_1 &= (r\lambda, \lambda; 0, \frac{1}{m}, 1), \\ \pi_2 &= (r^2\lambda, r\lambda, \lambda; 0, \frac{1}{m(1+k)}, \frac{1}{m}, 1), \\ \pi_3 &= (r^3\lambda, r^2\lambda, r\lambda, \lambda; 0, \frac{1}{m(1+k)^2}, \frac{1}{m(1+k)}, \frac{1}{m}, 1), \\ &\dots\dots\dots \\ \pi_s &= (r^s\lambda, r^{s-1}\lambda, \dots, r\lambda, \lambda; 0, \frac{1}{m(k+1)^{s-1}}, \frac{1}{m(k+1)^{s-2}}, \dots, \frac{1}{m(k+1)}, \frac{1}{m}, 1), \\ &\dots\dots\dots \end{aligned}$$

Recall section 4.1.2 and set $f := f_\alpha$. One further checks that $\pi_i = f^i \pi_\lambda$. Notice that the linear path $(r\lambda)t = (r\lambda; 0, 1)$ is not always a GLS path unlike the Kac-Moody case. One sees that $(r\lambda)t$ is a GLS path if and only if $m = 1$. Again one sees that $(r^s\lambda)t$ is a GLS path for all $s \in \mathbb{N}$, if and only if $m = 1$ and $k = 0$.

5.2.5. For all $\lambda \in P^+$ we denote by \mathbb{P}_λ the set of all GLS paths of shape λ . It is proven in [14] that when $I^{im} = \emptyset$ the set \mathbb{P}_λ is stable under the action of the root operators $f_i, e_i, i \in I$ defined in sections 4.1.2, 4.1.3 and $\mathbb{P}_\lambda = \mathcal{F}\pi_\lambda$, where π_λ is the linear path $\pi_\lambda(t) = \lambda t = (\lambda; 0, 1)$. In the language of (ref), \mathbb{P}_λ is a highest weight crystal.

Recall sections 3.1.6 and 4.2.1; the above imply that \mathbb{P}_λ is a strict subcrystal of \mathbb{P} . Furthermore, it is proven in [6] that \mathbb{P}_λ is isomorphic (as a crystal) to the crystal associated with the crystal basis of the (unique) highest weight module $V(\lambda)$ of highest weight λ over the quantized enveloping algebra of a Kac-Moody algebra \mathfrak{g} . Finally, by [15, Section 9] $\text{char } V(\lambda) = \text{char } \mathbb{P}_\lambda$.

Our aim is to prove analogous results in the generalized Kac-Moody case. However, this is not straightforward. Already \mathbb{P}_λ will not be a strict subcrystal of \mathbb{P} . This results in a number of complications, in particular to show that it is a highest weight crystal and with respect to the joining of paths (section 7.3). The latter is needed to prove that the $\mathbb{P}_\lambda, \lambda \in P^+$ form a closed family of highest weight crystals and as a consequence that this family is unique (section 8). The proof of the character formula (section 9) poses some particular challenges and is significantly more complicated.

5.3. Some integrality properties of the Generalized Lakshmibai-Seshadri paths.

In order to study the action of the operators $e_i, f_i, i \in I$ on the set of GLS paths \mathbb{P}_λ we need certain preliminary results which we give in this section.

5.3.1. Recall sections 4.1.1-4.1.4.

Lemma. Suppose that $\pi = (\lambda_1, \lambda_2, \dots, \lambda_s; 0, a_1, \dots, a_{s-1}, 1)$ is a Generalized Lakshmibai-Seshadri path of shape $\lambda \in P^+$ and let $i \in I^{im}$. Then the function h_i^π is increasing, $m_i^\pi = 0$ and one of the following is true :

- (1) $f_+^i(\pi) = 0$, $f_i\pi \neq 0$ and h_i^π is strictly increasing in a neighbourhood of 0,
- (2) $f_+^i(\pi) = 1$, $f_i\pi = 0$ and $h_i^\pi = 0$.

Moreover, $e_i\pi = 0$ if and only if $\alpha_i^\vee(\text{wt } \pi) < 1 - a_{ii}$.

Proof. Take $i \in I^{im}$; by lemma 2.2.1 and since the λ_j are in $T\lambda$, one has that $\alpha_i^\vee(\lambda_j) \geq 0$, for all j with $1 \leq j \leq s$. Since $\lambda_j > \lambda_{j+1}$, by lemma 5.1.4 we obtain $\alpha_i^\vee(\lambda_1) \geq \alpha_i^\vee(\lambda_2) \geq \dots \geq \alpha_i^\vee(\lambda_s) \geq 0$. Substitution in (10) shows that h_i^π is increasing in $[0, 1]$. Since $h_i^\pi(0) = 0$, we have that $m_i^\pi = 0$.

Moreover, either $\alpha_i^\vee(\lambda_1) > 0$, in which case h_i^π increases strictly in $[0, a_1]$, or $\alpha_i^\vee(\lambda_1) = 0$ and so $h_i^\pi(t) = 0$ for all $t \in [0, 1]$. In the first case $f_+^i(\pi) = 0$ and $f_i\pi \neq 0$. In the second case $f_+^i(\pi) = 1$ and so, by definition, $f_i\pi = 0$.

Since h_i^π is increasing and $m_i^\pi = 0$, one obtains $e_i\pi = 0$ if and only if $h_i^\pi(1) = \alpha_i^\vee(\text{wt } \pi) < 1 - a_{ii}$. \square

5.3.2. Lemma. Let $\nu := \nu_s \xleftarrow{\beta_s} \nu_{s-1} \xleftarrow{\beta_{s-1}} \dots \nu_1 \xleftarrow{\beta_1} \nu_0 =: \mu$ and take $\alpha_i \in \Pi_{re}$. If $r_i\mu < \mu$ and $r_i\nu \geq \nu$ or $r_i\mu \leq \mu$ and $r_i\nu > \nu$, then $\alpha_i = \beta_\ell$ for some ℓ , with $1 \leq \ell \leq s$ and $r_i\nu_t \geq \nu_t$, for all $t \geq \ell$.

Proof. Assume that $r_i\mu < \mu$ and $r_i\nu \geq \nu$ and recall that $\nu_t = r_{\beta_{t+1}}\nu_{t+1}$ for all t , with $0 \leq t \leq s-1$. By the hypothesis, there exists ℓ with $1 \leq \ell \leq s$ such that $r_i\nu_t \geq \nu_t$ for all $t \geq \ell$ and $r_i\nu_{\ell-1} < \nu_{\ell-1}$, so then $\alpha_i^\vee(\nu_{\ell-1}) < 0$. By lemma 5.1.3 (1) with $\nu = \nu_\ell$ and $\mu = \nu_{\ell-1}$, one has that $\text{dist}(r_i\nu_{\ell-1}, \nu_\ell) < \text{dist}(\nu_{\ell-1}, \nu_\ell) = 1$. This implies that $\nu_{\ell-1} = r_i\nu_\ell$ and $\alpha_i = \beta_\ell$. The second case follows similarly using lemma 5.1.3 (2). \square

5.3.3. The following lemma is similar to [15, Lemma 4.3]. We will give the proof in order to outline the fact that the real operators behave exactly as in the purely real case.

Lemma. Let $i \in I^{re}$ and let (μ, ν) be a pair of weights in $T\lambda$, with $\mu > \nu$.

- (1) If $r_i\mu < \mu$ and $r_i\nu \geq \nu$, then $\mu \geq r_i\nu$ and if there exists an a -chain for (μ, ν) , there exist one for the pair $(\mu, r_i\nu)$.
- (2) If $r_i\mu \leq \mu$ and $r_i\nu > \nu$, then $r_i\mu \geq \nu$ and if there exists an a -chain for (μ, ν) , there exist one for the pair $(r_i\mu, \nu)$.

Proof. Let

$$\nu := \nu_s \xleftarrow{\beta_s} \nu_{s-1} \xleftarrow{\beta_{s-1}} \dots \nu_1 \xleftarrow{\beta_1} \nu_0 =: \mu$$

be an a -chain for (μ, ν) and suppose that $r_i\mu < \mu$ and $r_i\nu \geq \nu$. By lemma 5.3.2 there exists ℓ , with $1 \leq \ell \leq s$ such that $\alpha_i = \beta_\ell$. We will prove that :

$$r_i\nu := r_i\nu_s \xleftarrow{\beta'_s} r_i\nu_{s-1} \xleftarrow{\beta'_{s-1}} \dots \xleftarrow{\beta'_{\ell+1}} r_i\nu_\ell = \nu_{\ell-1} \xleftarrow{\beta_{\ell-1}} \nu_{\ell-2} \dots \nu_1 \xleftarrow{\beta_1} \nu_0 =: \mu,$$

where $\beta'_t = r_i\beta_t$ for all t , with $\ell+1 \leq t \leq s$, is an a -chain.

First of all, since $r_{\beta'_t} = r_i r_{\beta_t} r_i$ we have that $r_{\beta'_t} r_i \nu_t = r_i r_{\beta_t} \nu_t = r_i \nu_{t-1}$. Again by lemma 5.3.2, $r_i \nu_t > \nu_t$ for all t with $\ell \leq t \leq s$, so lemma 5.1.3 (3) gives $\text{dist}(r_i \nu_{t-1}, r_i \nu_t) = \text{dist}(\nu_{t-1}, \nu_t) = 1$ for all t , with $\ell < t \leq s$. Finally $a_{\beta'_t}^\vee(r_i \nu_t) = a(r_i \beta_t)^\vee(r_i \nu_t) = a_{\beta_t}^\vee(\nu_t)$ and $\beta_t \in \Delta_{im}$ if and only if $\beta'_t \in \Delta_{im}$ which imply that the number $a_{\beta'_t}^\vee(r_i \nu_t)$ is an integer and is equal to 1 if β'_t is imaginary. The proof of the second statement is similar. \square

5.3.4. Corollary. *Let $i \in I^{re}$ and let (μ, ν) be a pair of weights in $T\lambda$ such that $\mu > \nu$.*

- (1) *If $r_i \mu > \mu$ and $r_i \nu \geq \nu$, then $r_i \mu \geq r_i \nu$ and if there exists an a -chain for (μ, ν) , there exist one for the pair $(r_i \mu, r_i \nu)$.*
- (2) *If $r_i \mu \leq \mu$ and $r_i \nu < \nu$, then $r_i \mu \leq r_i \nu$ and if there exists an a -chain for (μ, ν) , there exist one for the pair $(r_i \mu, r_i \nu)$.*

Proof. This follows by the proof of lemma 5.3.3. For example, to prove (1) take $\mu = \nu_\ell$ in the previous lemma. \square

5.3.5. Lemma. *Suppose $\mu, \nu \in T\lambda$ with $\mu > \nu$ and $i \in I^{im}$. If $\alpha_i^\vee(\mu) = \alpha_i^\vee(\nu)$, then $r_i \mu > r_i \nu$ and if there exists an a -chain for the pair (μ, ν) , then there exists one for $(r_i \mu, r_i \nu)$.*

Proof. Let

$$\nu := \nu_s \xleftarrow{\beta_s} \nu_{s-1} \xleftarrow{\beta_{s-1}} \cdots \nu_1 \xleftarrow{\beta_1} \nu_0 =: \mu$$

be an a -chain for (μ, ν) and $i \in I^{im}$. Suppose that $\alpha_i^\vee(\mu) = \alpha_i^\vee(\nu) \geq 0$. If equality holds, then $r_i \mu = \mu$, $r_i \nu = \nu$ and so there is nothing to prove. Thus we can assume $\alpha_i^\vee(\mu) = \alpha_i^\vee(\nu) > 0$. By lemma 5.1.4, r_i commutes with r_{β_j} for all j with $1 \leq j \leq s$. Hence

$$r_i \nu = r_i \nu_s \xleftarrow{\beta_s} r_i \nu_{s-1} \xleftarrow{\beta_{s-1}} \cdots r_i \nu_1 \xleftarrow{\beta_1} r_i \nu_0 = r_i \mu$$

is an a -chain for $(r_i \mu, r_i \nu)$. Indeed, $r_i \nu_{j-1} = r_{\beta_j} r_i \nu_j$ and $a_{\beta_j}^\vee(r_i \nu_j) = a_{\beta_j}^\vee(\nu_j)$ for all j with $1 \leq j \leq s$. Finally, $\text{dist}(r_i \nu_j, r_i \nu_{j+1}) = \text{dist}(\nu_j, \nu_{j+1}) = 1$ by lemma 5.1.5. \square

5.3.6. Call a path π *integral* if for all $i \in I$, the minimal value of the function h_i^π is an integer, i.e. $\min\{h_i^\pi(t) \mid t \in [0, 1]\} \in \mathbb{Z}$ for all $i \in I$. In other words, π is integral, if $m_i^\pi := \min\{h_i^\pi(t) \cap \mathbb{Z} \mid t \in [0, 1]\} = \min\{h_i^\pi(t) \mid t \in [0, 1]\}$. We will prove that a GLS path is integral. For this we need the following preliminary result :

Lemma. *Suppose that $\nu := \nu_s \xleftarrow{\beta_s} \nu_{s-1} \xleftarrow{\beta_{s-1}} \cdots \nu_1 \xleftarrow{\beta_1} \mu := \nu_0$ is an a -chain. For all t , $1 \leq t \leq s$ one has that $a_{\beta_t}^\vee(\mu) \in \mathbb{Z}$.*

Proof. We will prove the assertion by induction on $\text{dist}(\mu, \nu)$. First, if $\text{dist}(\mu, \nu) = 1$ and hence $\mu = r_{\beta_s} \nu$, the assertion follows by the definition of an a -chain. In the general case we have that $\mu = r_{\beta_1} r_{\beta_2} \cdots r_{\beta_s} \nu$. That $a_{\beta_1}^\vee(\mu) \in \mathbb{Z}$ follows again by definition. It remains to prove that $a_{\beta_i}^\vee(\mu) \in \mathbb{Z}$ for $i \neq 1$. Indeed, one has :

$$a_{\beta_i}^\vee(r_{\beta_1} r_{\beta_2} \cdots r_{\beta_s} \nu) = a_{\beta_i}^\vee(r_{\beta_2} \cdots r_{\beta_s} \nu) - a_{\beta_1}^\vee(r_{\beta_2} \cdots r_{\beta_s} \nu) \beta_i^\vee(\beta_1).$$

Notice that $a\beta_i^\vee(r_{\beta_2} \dots r_{\beta_s} \nu) \in \mathbb{Z}$ by the induction hypothesis to a strictly shorter a -chain in which ν_1 replaces ν_0 and since $a\beta_1^\vee(r_{\beta_2} \dots r_{\beta_s} \nu) \in \mathbb{Z}$ by the definition of an a -chain. Finally, of course $\beta_i^\vee(\beta_1) \in \mathbb{Z}$ and hence the result. \square

5.3.7. Let π be a GLS path. A key fact is that if h_i^π attains a local minimum at $t_0 \in [0, 1]$, then $h_i^\pi(t_0) \in \mathbb{Z}$. Since $\pi(0) = 0$ and $\pi(1) \in P$, it is enough to consider $t_0 \in]0, 1[$ and then by lemma 5.3.1 to take $i \in I^{re}$. We say that h_i^π attains a left local minimum at $t_0 \in]0, 1[$, if there exists $\varepsilon > 0$ such that h_i^π is strictly decreasing in $[t_0 - \varepsilon, t_0]$ and increasing in $[t_0, t_0 + \varepsilon]$. Presenting π as in equation (9) it is obvious that $t_0 = a_j$ for some j , with $0 < j < s$. Moreover, $\alpha_i^\vee(\lambda_j) < 0$ and $\alpha_i^\vee(\lambda_{j+1}) \geq 0$ (equivalently, $r_i \lambda_j < \lambda_j$ and $r_i \lambda_{j+1} \geq \lambda_{j+1}$).

Lemma. *Let π be a Generalized Lakshmibai-Seshadri path of shape λ and t_0 a left local minimum of h_i^π , for $i \in I^{re}$. Then $h_i^\pi(t_0) \in \mathbb{Z}$.*

Proof. By the first part of lemma 5.3.2 and the hypothesis one obtains $\alpha_i = \beta_t$ for some β_t in the a_j -chain for the pair $(\lambda_j, \lambda_{j+1})$. Then $a_j \alpha_i^\vee(\lambda_j) = a_j \beta_t^\vee(\lambda_j) \in \mathbb{Z}$, by lemma 5.3.6. Conclude by substituting into equations (8) and (10). \square

Remark. Define a right local minimum by shifting “strict” to the right in the definition of a left local minimum. Using the second part of lemma 5.3.2 we obtain as in lemma 5.3.7 above that if h_i^π , with $i \in I^{re}$, attains a right local minimum at $t_0 \in]0, 1[$, then $h_i^\pi(t_0) \in \mathbb{Z}$. In the sequel, a local minimum means a right or left local minimum.

5.3.8. The lemma below immediately follows by definition of a GLS path and equation (10) :

Lemma. *Let $\pi = (\lambda, \mathbf{a})$ be a Generalized Lakshmibai-Seshadri path and let $t_0 \in [0, 1]$ with $a_k < t_0 \leq a_{k+1}$ and $i \in I$. Then $h_i^\pi(t_0) \in \mathbb{Z}$ if and only if $t_0 \alpha_i^\vee(\lambda_{k+1}) \in \mathbb{Z}$. In particular, if $f_+^i(\pi) = a_j$ and $a_p < f_-^i(\pi) \leq a_{p+1}$ then $a_j \alpha_i^\vee(\lambda_{j+1}), f_-^i \alpha_i^\vee(\lambda_{p+1}) \in \mathbb{Z}$.*

5.3.9. Call a path $\pi(t)$ *monotone* if for all $i \in I$ such that $f_i \pi \neq 0$, the function h_i^π is strictly increasing in $[f_+^i(\pi), f_-^i(\pi)]$ and for all $t > f_-^i(\pi)$ one has that $h_i^\pi(t) \geq m_i^\pi + 1$.

Lemma. *A Generalized Lakshmibai-Seshadri path is monotone.*

Proof. By definition of $f_+^i(\pi), f_-^i(\pi)$ and by continuity, the function h_i^π does not attain integral values in the interval $]f_+^i(\pi), f_-^i(\pi)[$. If h_i^π is not increasing in $[f_+^i(\pi), f_-^i(\pi)]$ and since $h_i^\pi(f_+^i(\pi)) < h_i^\pi(f_-^i(\pi))$ it follows that h_i^π attains a local minimum at some $t_0 \in]f_+^i(\pi), f_-^i(\pi)[$. But then by lemma 5.3.7, $h_i^\pi(t_0) \in \mathbb{Z}$ which contradicts our first observation.

The second part is similar. We can assume $f_-^i(\pi) < 1$ and then $h_i^\pi(1) \geq m_i^\pi + 1 = h_i^\pi(f_-^i(\pi))$, by definition of $f_+^i(\pi)$. If $h_i^\pi(t) < m_i^\pi + 1$ for some $t \in]f_-^i(\pi), 1[$, we obtain a local minimum t_0 in this interval, with $h_i^\pi(t_0) \leq m_i^\pi + 1$, forcing $h_i^\pi(t_0) \leq m_i^\pi$ by integrality, and so contradicting the definition of $f_+^i(\pi)$. \square

6. THE CRYSTAL STRUCTURE OF THE SET OF GENERALIZED LAKSHMIBAI-SESHADRI PATHS

6.1. The action of the f_i , $i \in I$. In this section we will show that \mathbb{P}_λ is stable under the action of the root operators f_i , $i \in I$.

6.1.1. Let $\pi = (\boldsymbol{\lambda}, \mathbf{a}) = (\lambda_1, \lambda_2, \dots, \lambda_s; 0, a_1, \dots, a_{s-1}, a_s = 1)$ be a GLS path of shape λ , and recall sections 4.1.1, 4.1.2. In this section it is convenient for brevity to just suppose $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s$ and $0 = a_0 \leq a_1 \leq \dots \leq a_{s-1} \leq a_s = 1$. We recover strictness by just dropping some terms in the expression for π .

Since π is integral by lemma 5.3.7, $f_+^i(\pi) = a_t$ for some t , with $0 \leq t \leq s$. Let $f_i\pi \neq 0$; we can assume that $a_{p-1} < f_-^i(\pi) \leq a_p$ for $t+1 \leq p \leq s$. For simplicity in the the rest of this section we set $f_-^i := f_-^i(\pi)$ and $f_+^i := f_+^i(\pi)$.

6.1.2. Proposition. *Let $i \in I^{re}$ and π as above. The path $f_i\pi$ is equal to*

$$f_i\pi = (\lambda_1, \dots, \lambda_t, r_i\lambda_{t+1}, \dots, r_i\lambda_p, \lambda_p, \dots, \lambda_s; a_0, a_1, \dots, a_{p-1}, f_-^i, a_p, \dots, a_s),$$

and is a Generalized Lakshmibai-Seshadri path of shape λ . In particular, the set of Generalized Lakshmibai-Seshadri paths of shape λ is stable under the action of f_i , $i \in I^{re}$.

Proof. The proof of this proposition is exactly as in [15, Proposition 4.7], but we still give the proof for completeness. The fact that the resulting path is of the above form is clear. The only thing one has to check is that conditions (a), (b) of section 5.2.2 still hold for this new path. By monotonicity of π (see lemma 5.3.9) one has that $r_i\lambda_k > \lambda_k$ for all k , with $t < k \leq p$ and so by corollary 5.3.4 there exists an a_k -chain for $(r_i\lambda_k, r_i\lambda_{k+1})$ for all k , with $t < k < p$. On the other hand, $r_i\lambda_t \leq \lambda_t$ and lemma 5.3.3 implies that there exists an a_t -chain for $(\lambda_t, r_i\lambda_{t+1})$. Finally, since $h_i^\pi(f_-^i) \in \mathbb{N}$, there exists an f_-^i -chain for $(r_i\lambda_p, \lambda_p)$. \square

6.1.3. Let π be as in section 6.1.1 and take $i \in I^{im}$.

Proposition. *Let $i \in I^{im}$. For some $p \in \{1, 2, \dots, s\}$ the path $f_i\pi$ is equal to :*

$$f_i\pi = (r_i\lambda_1, \dots, r_i\lambda_p, \lambda_p, \dots, \lambda_s; a_0, a_1, \dots, a_{p-1}, f_-^i, a_p, \dots, a_s),$$

with $\alpha_i^\vee(\lambda_j) = \alpha_i^\vee(\lambda_{j+1})$ for all j , with $1 \leq j \leq p-1$ and is a Generalized Lakshmibai-Seshadri path. In particular, the set of Generalized Lakshmibai-Seshadri paths of shape λ is stable under the action of f_i , $i \in I^{im}$.

Proof. Assume that $f_i\pi \neq 0$ and so $f_+^i = a_0 = 0$. Choose p , with $1 \leq p \leq s$ such that $a_{p-1} < f_-^i \leq a_p$. By lemma 5.3.1 and the definition of f_-^i , the function h_i^π is strictly increasing in the interval $[0, f_-^i]$. Thus $r_i\lambda_p > \lambda_p \geq \lambda_{p+1}$ and so the resulting path will be of the above form. We need to prove that $f_i\pi$ is a GLS path. For this purpose it is sufficient to show the following :

- (1) For all j with $1 \leq j \leq p-1$ there exists an a_j -chain for the pair $(r_i\lambda_j, r_i\lambda_{j+1})$.
- (2) There exists an f_-^i -chain for the pair $(r_i\lambda_p, \lambda_p)$.

Recall equation (10); one has that

$$\pi(f_-^i) = \sum_{k=1}^{p-1} a_k(\lambda_k - \lambda_{k+1}) + f_-^i \lambda_p,$$

so that

$$1 = h_i^\pi(f_-^i) = \sum_{k=1}^{p-1} a_k \alpha_i^\vee(\lambda_k - \lambda_{k+1}) + f_-^i \alpha_i^\vee(\lambda_p).$$

Since $\lambda_k > \lambda_{k+1}$ for all k with $1 \leq k \leq p-1$, by lemma 5.1.4 we have that $\alpha_i^\vee(\lambda_k - \lambda_{k+1}) \geq 0$. On the other hand, $f_-^i \alpha_i^\vee(\lambda_p) \in \mathbb{Z}$ by lemma 5.3.8 and is strictly positive by monotonicity. This forces $\alpha_i^\vee(\lambda_k) = \alpha_i^\vee(\lambda_{k+1})$ for all k with $1 \leq k \leq p-1$. Hence by lemma 5.3.5 and since there exists an a_k -chain for $(\lambda_k, \lambda_{k+1})$, there exists one for $(r_i \lambda_k, r_i \lambda_{k+1})$ for all k with $1 \leq k \leq p-1$ and (1) follows.

Finally, by lemma 5.3.8 gives that $f_-^i \alpha_i^\vee(\lambda_p) \in \mathbb{Z}$ and so there exists an f_-^i -chain for $(r_i \lambda_p, \lambda_p)$ and (2) follows. \square

6.2. The action of the $e_i, i \in I$. In this section we study the action of the root operators $e_i, i \in I$ on the set \mathbb{P}_λ of GLS paths of shape λ .

6.2.1. Let π be as in section 6.1.1, $i \in I^e$ and recall section 4.1.3. By lemma 5.3.7, we have that $e_+^i := e_+^i(\pi) = a_k$ for some k with $1 \leq k \leq s$. Then $e_i \pi \neq 0$ if and only if $e_+^i > 0$. Let $e_i \pi \neq 0$, then we can assume that $e_-^i := e_-^i(\pi)$ is such that $a_{q-1} \leq e_-^i < a_q$, for some q with $0 \leq q \leq k$. The proof of the proposition below is similar to the proof of proposition 6.1.2 and so we omit it.

Proposition. *Assume that $i \in I^e$ and that $e_i \pi \neq 0$. Then $e_+^i > 0$ and hence the number $e_-^i \in [0, e_+^i]$ is defined. Let $e_+^i = a_k$ and $a_{q-1} \leq e_-^i < a_q$ for $1 \leq q < k$. Then the path $e_i \pi$ is equal to :*

$$e_i \pi = (\lambda_1, \dots, \lambda_q, r_i \lambda_q, \dots, r_i \lambda_k, \lambda_{k+1}, \dots, \lambda_s; a_0, \dots, a_{q-1}, e_-^i, a_q, \dots, a_s),$$

and is a Generalized Lakshmibai-Seshadri path of shape λ . In particular, the set of Generalized Lakshmibai-Seshadri paths of shape λ is stable under the action of $e_i, i \in I^e$.

6.2.2. Let $\pi \in \mathbb{P}_\lambda$ and $i \in I^{im}$. It can happen that $e_i \pi \in \mathbb{P} \setminus \mathbb{P}_\lambda$. Indeed, let λ be such that $\alpha_i^\vee(\lambda) \geq 1 - a_{ii}$. Recall that for all $\pi \in \mathbb{P}_\lambda$ one has that $\text{wt } \pi \in \lambda - Q^+$ and so $\alpha_i^\vee(\text{wt } \pi) \geq \alpha_i^\vee(\lambda) \geq 1 - a_{ii}$. Then by lemma 5.3.1 we obtain that $e_i \pi \neq 0$ for all $\pi \in \mathbb{P}_\lambda$. In particular, $e_i \pi_\lambda \neq 0$. But $\text{wt } e_i \pi_\lambda = \lambda + \alpha_i$ and so $e_i \pi \notin \mathbb{P}_\lambda$.

6.3. Crystal Structure of \mathbb{P}_λ .

6.3.1. For all $i \in I$ we set $e_i \pi = 0$ if and only if $e_i \pi \notin \mathbb{P}_\lambda$. For real indices, $e_i \pi \notin \mathbb{P}_\lambda$ is equivalent to $e_i \pi = 0$ in \mathbb{P} . Notice that this means that the “only if” of the last statement of lemma 5.3.1 will henceforth be violated. Recall the notation of section 5.2.5. Our aim is to show that $\mathbb{P}_\lambda = \mathcal{F} \pi_\lambda$.

6.3.2. Recall that \mathbb{P} has a crystal structure with crystal operations $\text{wt}, e_i, f_i, \varepsilon_i, \varphi_i$ for all $i \in I$ defined in section 4.2.1. Consider the embedding $\psi : \mathbb{P}_\lambda \hookrightarrow \mathbb{P}$. Then \mathbb{P}_λ is a subcrystal of \mathbb{P} . However, it is not a strict subcrystal of \mathbb{P} . Indeed, by propositions 6.1.2, 6.1.3 and 6.2.1, the map ψ commutes with all the crystal operations except the e_i , $i \in I^{im}$, though we still have $e_i\psi(\pi) = \psi(e_i\pi)$ if $e_i\pi \neq 0$, by definition 3.1.1 (4) and because f_i commutes with ψ .

6.3.3. Take $i \in I^{im}$. Recall that we have set $\varepsilon_i(\pi) = 0$ (section 4.2.1). By remark 5.2.3 (3) one has that $\text{wt } \pi \in \lambda - Q^+$ and so $\alpha_i^\vee(\text{wt } \pi) \geq 0$. Finally,

$$f_i\pi = 0 \Leftrightarrow f_+^i(\pi) = 1 \Leftrightarrow \alpha_i^\vee(\text{wt } \pi) = 0 \Leftrightarrow \varphi_i(\pi) = 0.$$

We conclude that $\mathbb{P}_\lambda \in \mathcal{B}$ (see section 3.1.4).

6.3.4. We will show that \mathbb{P}_λ is a highest weight crystal (proposition 6.3.5). For this we need the following preliminary lemma.

Given a reduced decomposition $w = r_{i_1}r_{i_2} \cdots r_{i_t}$ of $w \in W$, set $\text{Supp}(w) = \{\alpha_{i_k}, | 1 \leq k \leq t\}$. As is well-known it is independent of the choice of reduced decomposition.

Lemma. *Let $\mu, \nu \in T\lambda$ with $\lambda, \mu \in P^+$ and suppose that $\nu \xleftarrow{\beta} \mu$. Then β is a simple imaginary root.*

Proof. By hypothesis $\mu = r_\beta\nu$ for some $\beta \in W\Pi \cap \Delta^+$ with $\beta^\vee(\nu) > 0$. Then μ being dominant implies that $\beta \in W\Pi_{im}$.

Let $\beta = w\alpha_i$ and $i \in I^{im}$ and suppose that $w \notin \text{Stab}_W(\alpha_i)$. Then $\mu = r_\beta\nu = wr_iw^{-1}\nu$. By corollary 2.2.5, and since μ is dominant, every reduced expression of μ starts with r_j , $j \in I^{im}$. In particular, $w \in \text{Stab}_W(r_iw^{-1}\nu)$. Recalling that $r_iw^{-1}\nu$ is dominant, this by [7, Proposition 3.12] implies that for every root α_j in $\text{Supp}(w)$ one has that $\alpha_j^\vee(r_iw^{-1}\nu) = 0$. Then

$$\alpha_j^\vee(r_iw^{-1}\nu) = \alpha_j^\vee(w^{-1}\nu) - \alpha_i^\vee(w^{-1}\nu)a_{ji} = 0,$$

and since $\alpha_i^\vee(w^{-1}\nu) = \beta^\vee(\nu) > 0$ we must have that $\alpha_j^\vee(w^{-1}\nu) \leq 0$ for all $\alpha_j \in \text{Supp}(w)$.

Let $w = r_k w_1$, with $\ell(w) = \ell(w_1) + 1$. If $\alpha_k^\vee(\nu) = 0$ then $w^{-1}\nu = w_1^{-1}\nu$ and since $r_k\mu = \mu$ we can choose $\beta = w_1\alpha_i$.

In the above manner, we are reduced to the case where $\alpha_k^\vee(\nu) \neq 0$. Note that $w_1^{-1}\alpha_k \in \Delta^+$ and that we can write $(w_1^{-1}\alpha_k)^\vee = \sum_{\alpha_j \in \text{Supp}(w)} n_j \alpha_j^\vee$ with $n_j \geq 0$ for all j . Then :

$$\alpha_k^\vee(\nu) = (w^{-1}\alpha_k)^\vee(w^{-1}\nu) = -(w_1^{-1}\alpha_k)^\vee(w^{-1}\nu) = - \sum_{\alpha_j \in \text{Supp}(w)} n_j \alpha_j^\vee(w^{-1}\nu) \geq 0,$$

which by assumption on r_k forces $\alpha_k^\vee(\nu) > 0$ and so $r_k\nu > \nu$. Set $\beta_1 = w_1\alpha_i$, then $\beta_1^\vee(r_k\nu) = \alpha_i^\vee(w^{-1}\nu) > 0$ and consequently $r_{\beta_1}r_k\nu = w_1r_iw^{-1}\nu = \mu > r_k\nu$. We conclude that $\mu > r_k\nu > \nu$, which implies that $\text{dist}(\mu, \nu) \geq 2$ which contradicts our hypothesis. Hence $w = \text{id}$, mod $\text{Stab}_W(\alpha_i)$ and $\beta = \alpha_i \in \Pi_{im}$. \square

6.3.5. Let \mathcal{E} be the monoid generated by the $e_i, i \in I$ and set $\mathbb{P}_\lambda^\mathcal{E} = \{\pi \in \mathbb{P}_\lambda \mid e_i\pi = 0, \text{ for all } i \in I\}$.

Proposition. *Let $\pi \in \mathbb{P}_\lambda$. Then $e_i\pi = 0$ for all $i \in I$ if and only if $\pi = \pi_\lambda$, that is $\mathbb{P}_\lambda^\mathcal{E} = \{\pi_\lambda\}$. Moreover, $\mathbb{P}_\lambda = \mathcal{F}\pi_\lambda$.*

Proof. It is clear that $e_i\pi_\lambda = 0$ for all $i \in I$, since for all $\pi \in \mathbb{P}_\lambda$, one has that $\text{wt } \pi \prec \lambda$. Let

$$\pi = (\lambda_1, \lambda_2, \dots, \lambda_s : 0, a_1, \dots, a_{s-1}, a_s = 1)$$

be a path in \mathbb{P}_λ and notice that $\pi = \pi_\lambda$ if and only if $\lambda_1 = \lambda$. Suppose that $e_i\pi \notin \mathbb{P}_\lambda$, for all $i \in I$. Since for $i \in I^{re}$, the e_i preserve the set \mathbb{P}_λ our assumption implies that $e_i\pi = 0$ for all $i \in I^{re}$. This means that $\alpha_i^\vee(\lambda_1) \geq 0$, for all $i \in I^{re}$, that is λ_1 is dominant (and different from λ). On the other hand, by definition of a GLS path there exists an a_1 -chain

$$\lambda_2 := \nu_s \xleftarrow{\beta_s} \dots \nu_1 \xleftarrow{\beta_1} \nu_0 =: \lambda_1.$$

(If $a_1 = 1$ we set $\lambda_2 = \lambda$.) Then by lemma 6.3.4 we must have that $\beta_1 = \alpha_i$ for some $i \in I^{im}$. Hence $a_1\alpha_i^\vee(\nu_1) = 1$ and applying proposition 6.1.3 we have that

$$\pi' = (\nu_1, \lambda_2, \dots, \lambda_s; 0, a_1, \dots, a_s = 1)$$

is such that $f_i\pi' = \pi$ and so $e_i\pi \in \mathbb{P}_\lambda$. We conclude that the only path in \mathbb{P}_λ killed by all the $e_i, i \in I$ is π_λ .

We will prove now that $\mathbb{P}_\lambda = \mathcal{F}\pi_\lambda$. Since $\pi_\lambda = (\lambda; 0, 1) \in \mathbb{P}_\lambda$ and \mathbb{P}_λ is stable under the action of the $f_i, i \in I$ by propositions 6.1.2 and 6.1.3, one obtains $\mathcal{F}\pi_\lambda \subset \mathbb{P}_\lambda$. For the reverse inclusion it is enough by definition 3.1.1 (4) to show that $\pi_\lambda \in \mathcal{E}\pi$, for all $\pi \in \mathbb{P}_\lambda$. Since $\text{wt } \pi \in \lambda - Q^+$ and $\text{wt}(e_i\pi) = \text{wt } \pi + \alpha_i$, we obtain $\mathcal{E}\pi \cap \mathbb{P}_\lambda^\mathcal{E} \neq \emptyset$ and so the assertion follows from the first part. \square

7. CLOSED FAMILIES OF HIGHEST WEIGHT CRYSTALS

Call a family $\{B(\lambda) \mid \lambda \in P^+\}$ of highest weight crystals *closed under tensor products* or simply *closed* if for all $\lambda, \mu \in P^+$ the element $b_\lambda \otimes b_\mu$ of $B(\lambda) \otimes B(\mu)$ generates a crystal isomorphic to $B(\lambda + \mu)$. Our aim now is to prove that the family $\{\mathbb{P}_\lambda \mid \lambda \in P^+\}$ is closed.

Let $\lambda, \mu \in P^+$ and set $\nu := \lambda + \mu \in P^+$. We need to show that the crystals generated by $\pi_\lambda \otimes \pi_\mu$ and π_ν are isomorphic. As in [15], the proof involves deforming the path $\pi_\lambda \otimes \pi_\mu$ to π_ν without changing the crystal graph it generates. To do this we need to introduce some operations on \mathbb{P} . The fact that the crystals \mathbb{P}_λ and the crystal generated by $\pi_\lambda \otimes \pi_\mu$ are not strict subcrystals of \mathbb{P} causes some significant extra difficulty.

7.1. Deformations of paths.

7.1.1. *The join of two paths.* Let $s \leq s'$ be two rational numbers in $[0, 1]$, θ the trivial path defined by $\theta(t) = 0$ for all $t \in [0, 1]$ and let $\pi, \pi' \in \mathbb{P}$. Define $\pi * \theta_s^{s'} * \pi'$ to be the path:

$$(\pi * \theta_s^{s'} * \pi')(t) = \begin{cases} \pi(t), & t \in [0, s], \\ \pi(s), & t \in [s, s'], \\ \pi(s) + \pi'(t - s'), & t \in [s', 1]. \end{cases}$$

It is the concatenation of the truncated paths $\pi^s(t) : [0, s] \rightarrow \mathbb{Q}P$, $\pi'_{s'}(t) : [s', 1] \rightarrow \mathbb{Q}P$ and the trivial path θ . Clearly, if $s = s'$ and $\pi = \pi'$, then $\pi * \theta_s^{s'} * \pi' = \pi$. The reason for introducing this operation is explained in the section below.

7.1.2. Take $\lambda, \mu \in P^+$. We recall that by our conventions $[0, 1] \subset \mathbb{Q}$. Let $x \in [0, 1]$ and set $\pi^x = (1 - x)\pi_\lambda \otimes \pi_\mu + x\pi_\nu$. Then $\pi^x \in \mathbb{P}$ with $\text{wt } \pi^x = \nu$ for all $x \in [0, 1]$ and $\pi^0 = \pi_\lambda \otimes \pi_\mu$, $\pi^1 = \pi_\nu$. One can write $\pi^x = \pi_\delta \otimes \pi_{\delta'}$, where $\delta = (1 - x)\lambda + \frac{1}{2}x\nu$ and $\delta' = (1 - x)\mu + \frac{1}{2}x\nu$. Of course $\delta + \delta' = \nu$ but δ, δ' are not in general in the weight lattice and thus $\pi_\delta, \pi_{\delta'}$ are not in \mathbb{P} . However, one can find a positive integer r , such that $r\delta, r\delta' \in P$. Then $\pi^x = \pi_{r\delta} * \theta_{1/r}^{1-1/r} * \pi_{r\delta'}$ up to parametrization.

In section 7.2 we give sufficient conditions for any two paths π, π' to generate isomorphic crystals. Then, in sections 7.3 and 7.4, we show that the set of paths $\{\pi^x, x \in \mathbb{Q}\}$ satisfies these conditions, and in particular that $\mathcal{F}(\pi_\lambda \otimes \pi_\mu)$ is a highest weight crystal isomorphic to $\mathcal{F}\pi_\nu = \mathbb{P}_\nu$.

7.2. Distance of paths.

7.2.1. Let \mathcal{A} denote the monoid generated by the $e_i, f_i \in I$ and let $J \subset I$ be a finite subset of I . Denote by $\mathcal{A}_J, \mathcal{F}_J$ the monoids generated by the $e_i, f_i, i \in J$ and $f_i, i \in J$ respectively. Clearly, $\mathcal{A}_J \subset \mathcal{A}$ and $\mathcal{F}_J \subset \mathcal{F}$. Set $c_J = \max\{|a_{ij}|, i, j \in J\}$. For all $\pi, \pi' \in \mathbb{P}$, define their J -distance $d_J(\pi, \pi')$ to be :

$$d_J(\pi, \pi') = \max\{|\alpha_i^\vee(\pi(t) - \pi'(t))|, t \in [0, 1], i \in J\}.$$

7.2.2. The following lemma is the initial step in establishing the isomorphism theorem.

Lemma. *Let π, π' be integral and monotone paths such that $d_J(\pi, \pi') < \epsilon < 1$. Then for all $i \in J$ one has :*

- (1) $m_i^\pi = m_i^{\pi'}$ and $\text{wt } \pi = \text{wt } \pi'$.
- (2) If $f_i\pi \neq 0$, then $f_i\pi' \neq 0$ and $d_J(f_i\pi, f_i\pi') < 2c_J\epsilon$.
- (3) For $i \in I^{re} \cap J$, if $e_i\pi \neq 0$, then $e_i\pi' \neq 0$. If $e_i\pi, e_i\pi' \neq 0$ then $d_J(e_i\pi, e_i\pi') < 2c_J\epsilon$.

Proof. Statement (1) is an immediate consequence of the definitions and integrality. By section 4.2.1 and (1) we obtain $\varepsilon_i(\pi) = \varepsilon_i(\pi')$ and $\varphi_i(\pi) = \varphi_i(\pi')$ and thus the first part of (2) and (3) follow by normality for $i \in I^{re}$. For $i \in I^{im}$ the first part of (2) follows by (1) and section 6.3.3. The second part of (2) follows exactly as in [6, Lemma 6.4.25]. A key point is to show that the intervals $[f_+^i(\pi), f_-^i(\pi)]$ and $[f_+^i(\pi'), f_-^i(\pi')]$ have non-empty intersection. A similar comment applies to the second part of (3). \square

Remark. Notice that we do not obtain that $e_i\pi \neq 0$ implies $e_i\pi' \neq 0$ as it does for real indices since the “only if” of lemma 5.3.1 is violated (see section 6.3.1). This leads to an extra difficulty ultimately resolved by lemma 7.3.7.

7.3. Joining Generalized Lakshmibai-Seshadri paths. Throughout this section fix $\lambda, \mu \in P^+$. Let $\mathbb{P}_\lambda, \mathbb{P}_\mu$ be the sets of Generalized Lakshmibai-Seshadri of shape λ, μ respectively and recall section 7.1.1. We will join under certain conditions paths in \mathbb{P}_λ with paths in \mathbb{P}_μ . First we need the following preliminary result.

7.3.1. Lemma. *Let $\lambda, \mu \in P^+$. Then, for all $\tau \in T$ one has that $\beta^\vee(\tau\mu) > 0$ implies $\beta^\vee(\tau\lambda) \geq 0$, for all $\beta \in W\Pi \cap \Delta^+$.*

Proof. Since $\beta = w\alpha_i \in W\Pi$ one has $\beta^\vee(\tau\mu) = \alpha_i^\vee(w^{-1}\tau\mu)$ which reduces us to the case $\alpha_i \in \Pi$. For $\alpha_i \in \Pi_{im}$ one always has that $\alpha_i^\vee(\tau\lambda) \geq 0$, by lemma 2.2.1. Suppose that $\alpha_i \in \Pi_{re}$. Take $\tau \in T$; one can write $\tau = w_0\tau'$, where $\tau' = r_{i_1}w_1r_{i_2}\cdots r_{i_k}w_k$ is a dominant reduced expression of τ' . Then by lemma 2.2.3, one has that $\tau'\lambda, \tau'\mu \in P^+$.

Suppose that $\alpha_i^\vee(w_0\tau'\mu) > 0$, then $(w_0^{-1}\alpha_i)^\vee(\tau'\mu) > 0$, which implies that $w_0^{-1}\alpha_i \in \Delta^+$, since $\tau'\mu \in P^+$. Then $(w_0^{-1}\alpha_i)^\vee(\tau'\lambda) \geq 0$, since $\tau'\lambda \in P^+$ and so $\alpha_i^\vee(\tau\lambda) \geq 0$. \square

7.3.2. Let $\tau \in T$ and suppose that $\tau\mu > \mu$. By definition we may write $\tau\mu = r_{\beta_1}\cdots r_{\beta_s}\mu$ with $\beta_t^\vee(r_{\beta_{t+1}}\cdots r_{\beta_s}\mu) > 0$, for all t , with $1 \leq t \leq s$. By the previous lemma one has that $\beta_t^\vee(r_{\beta_{t+1}}\cdots r_{\beta_s}\lambda) \geq 0$ for all t and so $r_{\beta_t}\cdots r_{\beta_s}\lambda \geq r_{\beta_{t+1}}\cdots r_{\beta_s}\lambda$. In the expression $\tau\lambda = r_{\beta_1}\cdots r_{\beta_s}\lambda$, omit the r_{β_t} if $\beta_t^\vee(r_{\beta_{t+1}}\cdots r_{\beta_s}\lambda) = 0$, that is if $r_{\beta_t} \in \text{Stab}_T(r_{\beta_{t+1}}\cdots r_{\beta_s}\lambda)$, and denote by $\bar{\tau}$ the new element in T . One has $\bar{\tau}\lambda = \tau\lambda$ and $\bar{\tau}\lambda \geq \lambda$. Notice that if $\tau_1\tau_2\mu > \tau_2\mu > \mu$ then $\bar{\tau}_1\bar{\tau}_2\lambda = \overline{\tau_1\tau_2}\lambda \geq \bar{\tau}_2\lambda \geq \lambda$.

7.3.3. Definition. Fix two rational numbers $0 < s \leq s' < 1$ and let

$$\pi = (\lambda_1, \dots, \lambda_k; 0, a_1, \dots, a_{k-1}, 1) \in \mathbb{P}_\lambda, \pi' = (\mu_1, \dots, \mu_\ell; 0, b_1, \dots, b_{\ell-1}, 1) \in \mathbb{P}_\mu,$$

be such that $a_{k-1} < s \leq s' < b_1$. Observe that by equation (10), $\pi(t)$ is a translate of $(t - a_{k-1})\lambda_k$ in $[a_{k-1}, s]$ and $\pi'(t) = t\tau\mu$ in $[s', b_1]$.

We will assume that $\mu_t = r_{\beta_t}\mu_{t+1}$ with $\text{dist}(\mu_t, \mu_{t+1}) = 1$ for all t , with $1 \leq t \leq \ell - 1$ and $\mu_\ell = \mu$, by allowing $b_t = b_{t+1}$ if necessary. Set $\tau = r_{\beta_1}r_{\beta_2}\cdots r_{\beta_{\ell-1}}$. Then $\mu_1 = \tau\mu \geq \mu$. We say that the paths π, π' can be properly joined across $[s, s']$ if the following two conditions hold :

- (1) $\lambda_k \geq \bar{\tau}\lambda$ and if $\lambda_k > \bar{\tau}\lambda$ there exists an s -chain for the pair $(\lambda_k, \bar{\tau}\lambda)$.
- (2) For all t , with $1 \leq t \leq \ell - 1$, if $\beta_t \in \Delta_{im}$ one has that $s\beta_t^\vee(r_{\beta_{t+1}}\cdots r_{\beta_{\ell-1}}\lambda) < 1$.

We call (1) and (2) the joining conditions. Note that it is enough to consider the second condition for the roots β_t appearing in $\bar{\tau}$, that is $r_{\beta_t} \notin \text{Stab}_T(r_{\beta_{t+1}}\cdots r_{\beta_{\ell-1}}\lambda)$. We may write

$$(11) \quad \pi * \theta_s^{s'} * \pi' = (\lambda_1, \lambda_2, \dots, \lambda_k, \theta, \mu_1, \mu_2, \dots, \mu_\ell; 0, a_1, \dots, a_{k-1}, s, s', b_1, \dots, b_{\ell-1}, 1),$$

where we interpret the right hand side as a path using (9). We denote by $\mathbb{P}_\lambda * \theta_s^{s'} * \mathbb{P}_\mu$ the set of paths $\pi * \theta_s^{s'} * \pi'$ where $\pi \in \mathbb{P}_\lambda$, $\pi' \in \mathbb{P}_\mu$ can be properly joined across $[s, s']$. Of course if $\lambda = \mu$ and $s = s'$, the set $\mathbb{P}_\lambda * \theta_s^{s'} * \mathbb{P}_\mu$ is equal to \mathbb{P}_λ .

Remark. Let $\mu_\ell \xleftarrow{\beta_{\ell-1}} \mu_{\ell-1} \cdots \mu_2 \xleftarrow{\beta_1} \mu_1$ and suppose that the second joining condition holds with $\tau = r_{\beta_1} r_{\beta_2} \cdots r_{\beta_{\ell-1}}$ as specified above. Assume $i \in I^{re}$.

(1) If $r_i \mu_t > \mu_t$ for all t , with $1 \leq n \leq t \leq m \leq \ell$ and $r_i \mu_{n-1} \leq \mu_{n-1}$, then

$$\mu_\ell \xleftarrow{\beta_{\ell-1}} \mu_{\ell-1} \cdots \mu_m \xleftarrow{\alpha_i} r_i \mu_m \xleftarrow{r_i \beta_{m-1}} \cdots r_i \mu_{n+1} \xleftarrow{r_i \beta_n} r_i \mu_n = \mu_{n-1} \cdots \mu_2 \xleftarrow{\beta_1} \mu_1.$$

As above, this specifies the element $\tilde{\tau} = r_{\beta_1} \cdots r_{\beta_{n-2}} r_{\beta'_n} \cdots r_{\beta'_{m-1}} r_i r_{\beta_m} \cdots r_{\beta_{\ell-1}}$, where $\beta'_t = r_i \beta_t$, for all t , with $n \leq t \leq m-1$, relative to which the second joining condition holds because no new scalar products appear.

(2) If $r_i \mu_t < \mu_t$ for all t , with $1 \leq n \leq t \leq m \leq \ell$ and $r_i \mu_{m+1} \geq \mu_{m+1}$, then

$$\mu_\ell \xleftarrow{\beta_{\ell-1}} \mu_{\ell-1} \cdots \mu_{m+1} = r_i \mu_m \xleftarrow{r_i \beta_{m-1}} \cdots r_i \mu_{n+1} \xleftarrow{r_i \beta_n} r_i \mu_n \xleftarrow{\alpha_i} \mu_n \cdots \mu_2 \xleftarrow{\beta_1} \mu_1.$$

Similarly, relative to $\tilde{\tau} = r_{\beta_1} \cdots r_{\beta_{n-1}} r_i r_{\beta'_n} \cdots r_{\beta'_{m-1}} r_{\beta_{m+1}} \cdots r_{\beta_{\ell-1}}$, where $\beta'_t = r_i \beta_t$, for all t , with $n \leq t \leq m-1$, the second joining condition holds.

7.3.4. The subsets $\mathbb{P}_\lambda * \theta_s^{s'} * \mathbb{P}_\mu$ of \mathbb{P} are more general than the sets of Generalized Lakshmibai-Seshadri paths and they still have their nice properties as we show in the following lemmata. Recall (see section 5.3.6) what is meant by an integral path. We alter the definition of a monotone path (section 5.3.9) by requiring h_i^π to be increasing and not necessarily strictly increasing in $[f_i^+(\pi), f_i^-(\pi)]$.

Lemma. *A path $\pi \in \mathbb{P}_\lambda * \theta_s^{s'} * \mathbb{P}_\mu$ is integral and monotone.*

Proof. Let $\pi \in \mathbb{P}_\lambda$, $\pi' \in \mathbb{P}_\mu$, $s, s' \in [0, 1]$ be as in section 7.3.3 and assume that $\pi * \theta_s^{s'} * \pi' \in \mathbb{P}_\lambda * \theta_s^{s'} * \mathbb{P}_\mu$.

Set $h_i := h_i^{\pi * \theta_s^{s'} * \pi'}$ and $m_i := m_i^{\pi * \theta_s^{s'} * \pi'}$. Since the path $\pi * \theta_s^{s'} * \pi'$ is piecewise linear, a local minimum of the function h_i is attained at some a_x , $0 \leq x \leq k-1$ or b_y , $1 \leq y \leq \ell$ or at s, s' . If a local minimum of h_i is attained at $t \leq a_{k-1}$ or at $t \geq b_1$ then this number is an integer by lemma 5.3.7, since π, π' are Generalized Lakshmibai-Seshadri paths.

It remains to examine the case where $\min\{h_i(t) | t \in [0, 1]\} = h_i(s) = h_i(s')$. This will mean that $\alpha_i^\vee(\lambda_k) \leq 0$ and $\alpha_i^\vee(\mu_1) \geq 0$. If one of these numbers is zero, then $h_i(s) = h_i(a_r)$ for some r , $1 \leq r \leq k-1$ or $h_i(s) = h_i(b_{r'})$ for some r' , $1 \leq r' \leq \ell$ and is an integer.

Assume then that $\alpha_i^\vee(\lambda_k) < 0$ and $\alpha_i^\vee(\mu_1) > 0$. Since we have $\mu_1 = \tau\mu$, lemma 7.3.1 gives that $\alpha_i^\vee(\tau\lambda) = \alpha_i^\vee(\bar{\tau}\lambda) \geq 0$. Since there exists an s -chain for the pair $(\lambda_k, \bar{\tau}\lambda)$, lemma 5.3.3 (1) gives an s -chain for $(\lambda_k, r_i \bar{\tau}\lambda)$. If

$$\bar{\tau}\lambda := \nu_t \xleftarrow{\beta_t} \nu_{t-1} \cdots \nu_1 \xleftarrow{\beta_1} \nu_0 =: \lambda_k,$$

by lemma 5.3.2 one has that $\alpha_i = \beta_m$, for some m with $1 \leq m \leq t$ and then lemma 5.3.6 gives $s\alpha_i^\vee(\lambda_k) = s\beta_m^\vee(\lambda_k) \in \mathbb{Z}$. Yet $\pi * \theta_s^{s'} * \pi'(s) = \sum_{j=1}^k a_j(\lambda_j - \lambda_{j+1}) + s\lambda_k$ and since π_1 is a GLS path, remark 5.2.3 gives $a_j(\lambda_j - \lambda_{j+1}) \in Q$ for all j with $1 \leq j \leq k$. Hence $h_i(s) \in \mathbb{Z}$. We conclude that the path $\pi * \theta_s^{s'} * \pi'$ is integral.

Now if $f_i^-(\pi) < s$ or $f_i^+(\pi) > s'$ monotonicity follows by lemma 5.3.9. In the case where $f_i^+(\pi) \leq s \leq s' \leq f_i^-(\pi)$ the path is monotone in the weaker sense (since $h_i(s) = h_i(s')$). \square

7.3.5. Lemma. *Let $\lambda \in P^+$. The set $\mathbb{P}_\lambda * \theta_s^{s'} * \mathbb{P}_\mu$ is stable under the action of f_i , $i \in I$.*

Proof. Let $\pi * \theta_s^{s'} * \pi' \in \mathbb{P}_\lambda * \theta_s^{s'} * \mathbb{P}_\mu$ and write it as in (11). We will show that if $f_i(\pi * \theta_s^{s'} * \pi') \neq 0$ then $f_i(\pi * \theta_s^{s'} * \pi') \in \mathbb{P}_\lambda * \theta_s^{s'} * \mathbb{P}_\mu$.

In this proof we set $f_+^i := f_+^i(\pi * \theta_s^{s'} * \pi')$, $f_-^i := f_-^i(\pi * \theta_s^{s'} * \pi')$ and $h_i := h_i^{\pi * \theta_s^{s'} * \pi'}$. If $i \in I^{re}$ and $f_-^i < s$ or $f_+^i > s'$, then the first joining condition follows by proposition 6.1.2 and the second one trivially in the first case and by the remark in section 7.3.3 in the second case. On the other hand, if $i \in I^{im}$, then $f_+^i = 0$ or s' . It follows that the only cases which need to be checked are the following :

(1) Suppose that $f_-^i = s$; then

$f_i(\pi * \theta_s^{s'} * \pi') = (\lambda_1, \dots, \lambda_{t-1}, r_i \lambda_t, \dots, r_i \lambda_k, \theta, \mu_1, \dots, \mu_\ell; 0, a_1, \dots, a_{k-1}, s, s', b_1, \dots, b_{\ell-1}, 1)$, with $t = 1$, if $i \in I^{im}$. Again, the existence of an a_{t-1} -chain for $(\lambda_{t-1}, r_i \lambda_t)$ and of a_n -chains for the pairs $(r_i \lambda_n, r_i \lambda_{n+1})$ for all n with $1 \leq n \leq k-1$ follows as in propositions 6.1.2, 6.1.3. Now since $h_i(s) = m_i + 1 \in \mathbb{Z}$, by lemma 5.3.8 one has that $s\alpha_i^\vee(\lambda_k) \in \mathbb{N}^+$ and so there exists an s -chain for $(r_i \lambda_k, \lambda_k)$. Combined with the given chain for $(\lambda_k, \bar{\tau}\lambda)$, we obtain an s -chain for $(r_i \lambda_k, \bar{\tau}\lambda)$. We conclude that $f_i(\pi * \theta_s^{s'} * \pi') = \pi_1 * \theta_s^{s'} * \pi'$, where π_1 is the path given by

$$(12) \quad \pi_1 = (\lambda_1, \dots, \lambda_{t-1}, r_i \lambda_t, \dots, r_i \lambda_k, \lambda_k; 0, a_1, \dots, a_{k-1}, s, 1),$$

with $t = 1$ if $i \in I^{im}$ and π_1, π' can be joined across $[s, s']$. Since here τ is unchanged, the second joining condition immediately follows from the second condition on the starting path.

(2) Suppose now that $f_+^i < s$ and $f_-^i > s'$ and say $b_{m-1} < f_-^i \leq b_m$, with $1 \leq m \leq \ell$. Then

$$f_i(\pi * \theta_s^{s'} * \pi') = (\lambda_1, \dots, \lambda_{t-1}, r_i \lambda_t, \dots, r_i \lambda_k, \theta, r_i \mu_1, \dots, r_i \mu_m, \mu_m, \dots, \mu_\ell; 0, a_1, \dots, a_{k-1}, s, s', b_1, \dots, b_{m-1}, f_-^i, b_m, \dots, b_{\ell-1}, 1),$$

with $t = 1$, if $i \in I^{im}$. One has that $\alpha_i^\vee(\lambda_k) > 0$ and $\alpha_i^\vee(\mu_1) > 0$. We will show that there exists an s -chain for $(r_i \lambda_k, \bar{\tau}_i \bar{\tau} \lambda)$.

Suppose that $i \in I^{im}$. Recall that there exists an s -chain for $(\lambda_k, \bar{\tau}\lambda)$. By equation (8) we obtain $s(\lambda_k - \bar{\tau}\lambda) \in -Q^+$ and so $s\alpha_i^\vee(\lambda_k - \bar{\tau}\lambda) \in \mathbb{N}$, by lemma 2.1.11 (2). Yet $s\alpha_i^\vee(\bar{\tau}\lambda) \geq 0$, by lemma 2.2.1, whilst $s\alpha_i^\vee(\lambda_k) < 1$, since $f_-^i > s$, that is :

$$(13) \quad 0 \leq s\alpha_i^\vee(\bar{\tau}\lambda) \leq s\alpha_i^\vee(\lambda_k) < 1.$$

This forces $\alpha_i^\vee(\lambda_k) = \alpha_i^\vee(\bar{\tau}\lambda)$. By lemma 5.3.5 there exists an s -chain for $(r_i\lambda_k, r_i\bar{\tau}\lambda)$ and hence for $(r_i\lambda_k, \bar{r}_i\bar{\tau}\lambda)$.

Recall that $\mu_\ell \xleftarrow{\beta_{\ell-1}} \mu_{\ell-1} \cdots \mu_2 \xleftarrow{\beta_1} \mu_1$. Observe from proposition 6.1.3 that $f_i(\pi * \theta_s^{s'} * \pi')$ being specified as above means that $\alpha_i^\vee(\beta_t) = 0$ for all t , with $1 \leq t \leq m$ and $\mu_\ell \xleftarrow{\beta_{\ell-1}} \mu_{\ell-1} \cdots \mu_m \xleftarrow{\alpha_i} r_i\mu_m \xleftarrow{\beta_{m-1}} r_i\mu_{m-1} \cdots r_i\mu_2 \xleftarrow{\beta_1} r_i\mu_1$. Then $r_i\tau = r_{\beta_1} \cdots r_{\beta_m} r_i r_{\beta_{m+1}} \cdots r_{\beta_{\ell-1}}$ and the second joining condition reduces to $s\alpha_i^\vee(\bar{\tau}\lambda) < 1$, verified in (13).

Suppose that $i \in I^{re}$. Since $\alpha_i^\vee(\tau\mu) > 0$, we obtain by lemma 7.3.1 that $\alpha_i^\vee(\bar{\tau}\lambda) \geq 0$. Again $\alpha_i^\vee(\lambda_k) > 0$ and so there exists an s -chain for $(r_i\lambda_k, \bar{r}_i\bar{\tau}\lambda)$ by corollary 5.3.4 (1). The second joining condition in this case follows by the remark of section 7.3.3.

We conclude that $f_i(\pi * \theta_s^{s'} * \pi') = \pi_1 * \theta_s^{s'} * \pi'_2$ with

$$\pi_1 = (\lambda_1, \dots, \lambda_{t-1}, r_i\lambda_t, \dots, r_i\lambda_k, \lambda_k; 0, a_1, a_{k-1}, a, 1),$$

(where if $i \in I^{im}$, $t = 1$ and $a \in]s, 1]$ is such that $a\alpha_i^\vee(\lambda_k) = 1$, and if $i \in I^{re}$, $a = 1$),

$$(14) \quad \pi'_2 = (r_i\mu_1, \dots, r_i\mu_m, \mu_m, \dots, \mu_\ell; 0, b_1, \dots, b_{m-1}, f_-^i, b_m, \dots, b_{\ell-1}, 1),$$

and π_1, π'_2 can be joined across $[s, s']$.

(3) Finally suppose that $f_+^i = s' < f_-^i$. Then

$$\begin{aligned} f_i(\pi * \theta_s^{s'} * \pi') &= (\lambda_1, \dots, \lambda_k, \theta, r_i\mu, \dots, r_i\mu_m, \mu_m, \dots, \mu_\ell; \\ &\quad 0, a_1, \dots, a_{k-1}, s, s', b_1, \dots, b_{m-1}, f_-^i, b_m, \dots, b_{\ell-1}, 1). \end{aligned}$$

Suppose that $i \in I^{im}$. Then, we will have that $\alpha_i^\vee(\lambda_k) = 0$ and so $\alpha_i^\vee(\bar{\tau}\lambda) = 0$ and $\alpha_i^\vee(\mu_1) > 0$. But then $\bar{r}_i\bar{\tau}\lambda = \bar{\tau}\lambda$ so there exists an s -chain for $(\lambda_k, \bar{r}_i\bar{\tau}\lambda)$. The second joining condition follows by the vanishing of $\alpha_i^\vee(\bar{\tau}\lambda)$.

Suppose now that $i \in I^{re}$. We have that $\alpha_i^\vee(\lambda_k) \leq 0$ and $\alpha_i^\vee(\mu_1) = \alpha_i^\vee(\tau\mu) > 0$. Lemma 7.3.1 gives $\alpha_i^\vee(\bar{\tau}\lambda) \geq 0$ and so by lemma 5.3.3 there exists an s -chain for $(\lambda_k, \bar{r}_i\bar{\tau}\lambda)$, hence the first joining condition holds. The second one follows by the remark of section 7.3.3. Then $f_i(\pi * \theta_s^{s'} * \pi') = \pi * \theta_s^{s'} * \pi'_2$, where π'_2 is as in (14). The assertion follows. \square

7.3.6. Our aim now is to prove that $\mathcal{A}(\pi_\lambda \otimes \pi_\mu)$ is a highest weight crystal generated by $\pi_\lambda \otimes \pi_\mu$ over \mathcal{F} (see lemma 7.3.7). The following lemma is a preliminary result for this purpose.

Lemma. *Let $\nu := \nu_s \xleftarrow{\beta_s} \nu_{s-1} \xleftarrow{\beta_{s-1}} \cdots \xleftarrow{\beta_2} \nu_1 \xleftarrow{\beta_1} \nu_0 =: \mu$ be an a -chain for (μ, ν) , such that $\beta_l = w\alpha_i$, where $i \in I^{im}$ for some l , with $1 \leq l \leq s$. Suppose further that $a\alpha_i^\vee(\mu) = 1 - a_{ii}$. Then $\beta_l = \alpha_i$, $\mu = r_i\mu'$ and there exists an a -chain for (μ', ν) .*

Proof. Suppose that $\beta_l = w\alpha_i \neq \alpha_i$. Since $-\alpha_i$ is dominant, we obtain $\beta_l = \alpha_i + \beta \in \alpha_i + \text{NII}_{re}$ and $\alpha_i^\vee(\beta) \leq -1$. By the hypothesis,

$$(15) \quad a\alpha_i^\vee(\mu) = 1 - a_{ii}.$$

On the other hand,

$$(16) \quad a\alpha_i^\vee(\mu) = a\alpha_i^\vee(\nu_l) - a\beta_l^\vee(\nu_l)\alpha_i^\vee(\beta_l) - \sum_{q=1}^{l-1} a\beta_q^\vee(\nu_q)\alpha_i^\vee(\beta_q).$$

Now $a\beta_l^\vee(\nu_l) = 1$ by 5.2.2 (b), whereas $\alpha_i^\vee(\beta_l) = \alpha_i^\vee(\alpha_i + \beta) \leq a_{ii} - 1$. Then (15) and (16) give that :

$$(17) \quad 0 \geq \underbrace{a\alpha_i^\vee(\nu_l)}_{\geq 0} - \sum_{q=1}^{l-1} \underbrace{a\beta_q^\vee(\nu_q)\alpha_i^\vee(\beta_q)}_{\leq 0},$$

which means that all the summands in (17) are equal to zero so $\alpha_i^\vee(\nu_l) = 0$ and $\alpha_i^\vee(\beta_l) = a_{ii} - 1$. This on one hand means that $\alpha_i^\vee(\beta_q) = 0$ for all q , with $1 \leq q \leq l-1$ and so r_i commutes with all r_{β_q} with $1 \leq q \leq l-1$. On the other hand we can write $\beta_l = w\alpha_i = w_1 r_k \alpha_i = w_1(\alpha_i - \alpha_k^\vee(\alpha_i)\alpha_k)$, with $w_1 \alpha_k = \alpha_k + \beta_1$ and moreover $w_i \alpha_i = \alpha_i + \beta_2$, with $\beta_1, \beta_2 \in \mathbb{N}\Pi_{re}$. Yet $\alpha_i^\vee(\beta_l) = a_{ii} - 1$, which forces $\alpha_i^\vee(\beta_1) = 0$, $\alpha_i^\vee(\beta_2) = 0$ and $\alpha_i^\vee(\alpha_k)\alpha_k^\vee(\alpha_i) = 1$. The second condition forces $w_1 \alpha_i = \alpha_i$ and $\alpha_i^\vee(\alpha_k) = \alpha_k^\vee(\alpha_i) = -1$. Then $\gamma := w_1 \alpha_k$ is such that $\beta_l = w\alpha_i = r_\gamma \alpha_i$. Note that $\gamma^\vee(\alpha_i) = \alpha_k^\vee(\alpha_i) = -1$ and $\alpha_i^\vee(\gamma) = \alpha_i^\vee(\alpha_k) = -1$. Also $\beta_l = w_1 r_k \alpha_i = \alpha_i + \gamma$. Then

$$(18) \quad 1 = a\beta_l^\vee(\nu_l) = a\alpha_i^\vee(r_\gamma \nu_l) = -a\gamma^\vee(\nu_l)\alpha_i^\vee(\gamma) = a\gamma^\vee(\nu_l),$$

since $\alpha_i^\vee(\nu_l) = 0$ and $\alpha_i^\vee(\gamma) = a_{ik} = -1$. Again,

$$(19) \quad a(r_i r_\gamma \nu_l - \nu_l) = -a\gamma^\vee(\nu_l)r_i \gamma = -(\gamma + \alpha_i) = -\beta_l,$$

whilst

$$a(r_{\beta_l} \nu_l - \nu_l) = -\beta_l.$$

Then $\nu_{l-1} = r_{\beta_l} \nu_l = r_i r_\gamma \nu_l > r_\gamma \nu_l > \nu_l$ and so $\text{dist}(\nu_{l-1}, \nu_l) \geq 2$, which contradicts our hypothesis. Then necessarily $\beta_l = \alpha_i$ and r_i commutes with r_{β_q} for all q , with $1 \leq q < l$. It then follows that $\nu := \nu_s \xleftarrow{\beta_s} \nu_{s-1} \xleftarrow{\beta_{s-1}} \dots \nu_{l+1} \xleftarrow{\beta_{l+1}} \nu'_l \xleftarrow{\beta_{l-1}} \nu'_{l-1} \dots \xleftarrow{\beta_1} \nu'_1 \xleftarrow{\alpha_1} \nu_0 =: \mu$ is an a -chain, where $r_i \nu'_q = \nu_q$ for all q , with $1 \leq q \leq l$ and so there exists an a -chain for (ν'_1, ν) . \square

7.3.7. Lemma. *Let $\lambda, \mu \in P^+$. A path in $\mathcal{A}(\pi_\lambda \otimes \pi_\mu)$ is integral and monotone and the only path killed by the e_i , $i \in I$ is $\pi_\lambda \otimes \pi_\mu$. In particular, $\mathcal{A}(\pi_\lambda \otimes \pi_\mu) = \mathcal{F}(\pi_\lambda \otimes \pi_\mu)$.*

Proof. We can write $\pi_\lambda \otimes \pi_\mu = \pi_{2\lambda} * \theta_{1/2}^{1/2} * \pi_{2\mu} \in \mathbb{P}_{2\lambda} * \theta_{1/2}^{1/2} * \mathbb{P}_{2\mu}$, since the joining conditions become trivial. One has $\mathbb{P}_{2\lambda} * \theta_{1/2}^{1/2} * \mathbb{P}_{2\mu} \subset \mathbb{P}_\lambda \otimes \mathbb{P}_\mu$, but this inclusion in general is strict, since terms in the left hand side must satisfy the joining conditions.

By lemma 7.3.5 the set $\mathbb{P}_{2\lambda} * \theta_{1/2}^{1/2} * \mathbb{P}_{2\mu}$ is stable under the action of f_i , $i \in I$. We will show that as a subset of $\mathbb{P}_\lambda \otimes \mathbb{P}_\mu$ it is stable under the action of e_i , $i \in I$ and that the only path in $\mathbb{P}_{2\lambda} * \theta_{1/2}^{1/2} * \mathbb{P}_{2\mu}$ killed by all the e_i , $i \in I$ is the path $\pi_\lambda \otimes \pi_\mu$. This will give $\mathbb{P}_{2\lambda} * \theta_{1/2}^{1/2} * \mathbb{P}_{2\mu} = \mathcal{A}(\pi_\lambda \otimes \pi_\mu) = \mathcal{F}(\pi_\lambda \otimes \pi_\mu)$. Finally the integrality and monotonicity of the paths in $\mathcal{F}(\pi_\lambda \otimes \pi_\mu)$ will follow by lemma 7.3.4.

We first show that $\mathbb{P}_{2\lambda} * \theta_{1/2}^{1/2} * \mathbb{P}_{2\mu}$ is e_i stable in the above sense for all $i \in I$. Let $\tilde{\pi} = \pi * \theta_{1/2}^{1/2} * \pi'$ be as in paragraph 7.3.3. We will show that if $e_i \tilde{\pi} \neq 0$, then $e_i \tilde{\pi} \in \mathbb{P}_{2\lambda} * \theta_{1/2}^{1/2} * \mathbb{P}_{2\mu}$. Identify $\mathbb{P}_{2\lambda} * \theta_{1/2}^{1/2} * \mathbb{P}_{2\mu}$ with its image in $\mathbb{P}_\lambda \otimes \mathbb{P}_\mu$ and let $\tilde{\pi} = \pi_1 \otimes \pi_2$ with this identification.

Since $\tilde{\pi}(1/2) \in P$, we can only have either $e_i(\pi_1 \otimes \pi_2) = (e_i \pi_1) \otimes \pi_2$ or $e_i(\pi_1 \otimes \pi_2) = \pi_1 \otimes (e_i \pi_2)$ the choice depending on the crystal tensor product rules (section 3.2.1). It remains to check the joining condition for the new paths.

(1) Suppose first that $e_i(\pi_1 \otimes \pi_2) = (e_i \pi_1) \otimes \pi_2 \neq 0$.

If $e_+^i(\pi_1) < 1$ (and so $e_+^i(\tilde{\pi}) < 1/2$), λ_k, μ_1 are unchanged, so there is nothing to check. Suppose then that $e_+^i(\pi_1) = 1$, equivalently $e_+^i(\tilde{\pi}) = 1/2$.

Since μ_1 is unchanged we only need to show that there exists an $1/2$ -chain for $(2r_i \lambda_k, \bar{\tau} 2\lambda)$. This is equivalent to the existence of an 1 -chain for $(r_i \lambda_k, \bar{\tau} \lambda)$.

Let $i \in I^{re}$.

By the definition of $e_+^i(\tilde{\pi})$, one has that $\alpha_i^\vee(\lambda_k) < 0$ and $\alpha_i^\vee(\tau\mu) \geq 0$. If $\alpha_i^\vee(\tau\mu) > 0$, then by lemma 7.3.1, we have that $\alpha_i^\vee(\bar{\tau}\lambda) = \alpha_i^\vee(\tau\lambda) \geq 0$ and so by lemma 5.3.3 there exists an 1 -chain for $(\lambda_k, \bar{\tau}_i \bar{\tau} \lambda)$, since there exists one for $(\lambda_k, \bar{\tau} \lambda)$. If finally $\alpha_i^\vee(\tau\mu) = 0$, then $r_i \tau\mu = \tau\mu$ and the assertion is trivial.

Let now $i \in I^{im}$.

This means that there exists a path $\pi'_1 \in \mathbb{P}_\lambda$ with $\pi'_1 = (\lambda'_1, \dots, \lambda'_k; 0, a_1, \dots, a_{k-1}, 1)$ such that $f_i \pi'_1 = \pi_1$, which in turn gives $\lambda_t = r_i \lambda'_t$ for all t , with $1 \leq t \leq k$. Let

$$\bar{\tau} \lambda := \nu_s \xleftarrow{\beta_s} \nu_{s-1} \xleftarrow{\beta_{s-1}} \dots \xleftarrow{\beta_2} \nu_1 \xleftarrow{\beta_1} \nu_0 =: \lambda_k,$$

with $\beta_j \in W\Pi \cap \Delta^+$ for all j , with $1 \leq j \leq s$, be an 1 -chain for $(\lambda_k, \bar{\tau} \lambda)$. We need to show that there also exists an 1 -chain for $(\lambda'_k, \bar{\tau} \lambda)$. By the second joining condition, if $\bar{\tau} = r_{\beta_1} \dots r_{\beta_n}$, then $\beta_t^\vee(r_{\beta_{t+1}} \dots r_{\beta_n} \lambda) < 1$ for all t such that $\beta_t \in \Delta_{im}$. This forces $\bar{\tau} \in W$ which in turn implies that $\beta_l = w \alpha_i$ for some l , with $1 \leq l \leq s$ and some $w \in W$. By assumption that $e_+^i(\pi_1) = 1$, we obtain $\alpha_i^\vee(\lambda_k) = 1 - a_{ii}$. Then the assertion follows by lemma 7.3.6.

(2) Suppose that $e_i(\pi_1 \otimes \pi_2) = \pi_1 \otimes (e_i \pi_2) \neq 0$.

If $e_-^i(\pi_1) > 0$ (and so $e_-^i(\tilde{\pi}) > 1/2$), then since the λ_k, μ_1 are unchanged, the first joining condition is trivial. The second one follows by the remark of section 7.3.3.

Suppose then that $e_-^i(\pi_2) = 0$ and so $e_-^i(\tilde{\pi}) = 1/2$.

Suppose that $i \in I^{re}$.

We have that $\alpha_i^\vee(\lambda_k) \leq 0$ and $\alpha_i^\vee(\tau\mu) < 0$, which implies that $\alpha_i^\vee(\bar{\tau}\lambda) \leq 0$. If the latter is zero, the first joining condition trivially follows. If not, then $\bar{r}_i \bar{\tau} \lambda \xleftarrow{\alpha_i} \bar{\tau} \lambda$ and so there exists

an 1-chain for $(\lambda_k, \overline{r_i \tau} \lambda)$. The second joining condition follows by the remark of section 7.3.3. Suppose now that $i \in I^{im}$.

Then $\alpha_i^\vee(\lambda_k) = 0$, and so since $\lambda_k \geq \overline{\lambda}$, by lemma 5.1.4 we obtain that $\alpha_i^\vee(\lambda_k) = \alpha_i^\vee(\overline{\tau} \lambda) = 0$. It follows that the second joining condition holds and $\overline{r_i \tau} \lambda = \overline{\tau} \lambda$, hence there exists an 1-chain for $(\lambda_k, \overline{r_i \tau} \lambda)$. We conclude that $\pi_1, e_i \pi_2$ can be properly joined.

We finally show that $\pi_\lambda \otimes \pi_\mu$ is the only path in $\mathbb{P}_{2\lambda} * \theta_{1/2}^{1/2} * \mathbb{P}_{2\mu}$ killed by e_i , $i \in I$. For this we first show that every $\pi \in \mathbb{P}_\lambda \otimes \mathbb{P}_\mu$, killed by the e_i , $i \in I$, takes the form $\pi = \pi_\lambda \otimes \pi$, with $\pi \in \mathbb{P}_\mu$ and $\lambda + \pi(1) \in P^+$.

Recall the tensor product crystal operations of section 3.2.1. Take $\pi = \pi_1 \otimes \pi_2 \in \mathbb{P}_\lambda \otimes \mathbb{P}_\mu$ and assume that $e_i \pi = 0$ for all $i \in I$. Let $i \in I^{re}$. If $\varepsilon_i(\pi_2) > \varphi_i(\pi_1)$ one has that $e_i(\pi_1 \otimes \pi_2) = \pi_1 \otimes e_i \pi_2 \neq 0$ by normality. So we must have

$$(20) \quad \varphi_i(\pi_1) \geq \varepsilon_i(\pi_2), \quad \text{for all } i \in I^{re}.$$

But then $e_i \pi = e_i \pi_1 \otimes \pi_2$ and consequently, we must have $e_i \pi_1 = 0$, for all $i \in I^{re}$. Now take $i \in I^{im}$ and recall lemma 5.3.1. One has that $\varphi_i(\pi_1) > -a_{ii} \Leftrightarrow \alpha_i^\vee(\text{wt } \pi_1) \geq 1 - a_{ii} \Leftrightarrow e_i \pi = e_i \pi_1 \otimes \pi_2$. On the other hand, if $\alpha_i^\vee(\pi_1) < 1 - a_{ii}$, then $e_i \pi_1 = 0$ again by lemma 5.3.1. In both cases $e_i \pi_1 = 0$. We conclude that $e_i \pi = 0$ for all $i \in I$, only if $e_i \pi_1 = 0$ for all $i \in I$ which forces $\pi_1 = \pi_\lambda$. Notice also by (20) one has that

$$(21) \quad \alpha_i^\vee(\lambda + \text{wt } \pi_2) \geq 0, \quad \text{for all } i \in I,$$

that is $\lambda + \pi_2(1) \in P^+$. Set $J = \{i \in I \mid \alpha_i^\vee(\lambda) = 0\}$ and assume that $e_i(\pi_\lambda \otimes \pi) = 0$ for all $i \in I$. Then $e_i \pi = 0$ for all $i \in J$. A path in $\mathbb{P}_{2\lambda} * \theta_{1/2}^{1/2} * \mathbb{P}_{2\mu}$ killed by all the e_i , $i \in I$ will then be of the form $\tilde{\pi} = \pi_{2\lambda} * \theta_{1/2}^{1/2} * \pi$, where

$$\pi = (\mu_1, \dots, \mu_\ell; 0, b_1, \dots, b_{\ell-1}, 1)$$

is a GLS path of shape μ with $1/2 < b_1$ and $\mu_1 = \tau \mu$. Now the first joining condition forces $\tau \in \text{Stab}_T(\lambda)$. If we set $\mu_r = \tau_r \mu$ with $1 \leq r \leq \ell$, then since $\mu_r > \mu_{r+1}$ we will have that $\tau_r \in \text{Stab}_T(\lambda)$ for all r with $1 \leq r \leq \ell$. Take $i \in I \setminus J$. Then since $\text{Stab}_T(\lambda) = \langle r_i \mid i \in J \rangle$ by lemma 2.2.2, we have that $\text{wt } \pi \in \mu - \sum_{j \in J} \mathbb{N} \alpha_j$ and so $e_i \pi = 0$, for $i \in I \setminus J$ since the

set of weights of \mathbb{P}_μ lies in $\mu - Q^+$. Combined with our previous result, namely that $e_i \pi = 0$ for all $i \in J$, this forces $\mu_1 = \mu$ and the only path in $\mathbb{P}_{2\lambda} * \theta_{1/2}^{1/2} * \mathbb{P}_{2\mu}$ annihilated by all the e_i is $\pi_{2\lambda} * \theta_{1/2}^{1/2} * \pi_{2\mu} = \pi_\lambda \otimes \pi_\mu$. \square

7.4. The isomorphism theorem. Recall the family $\{\pi^x, x \in [0, 1]\}$ of section 7.1.2, which deforms the path $\pi_\lambda \otimes \pi_\mu$ to π_ν . We will show that $\mathcal{F}\pi^0 \simeq \mathcal{F}\pi^1$ and then that $\pi_\lambda \otimes \pi_\mu$ and π_ν generate isomorphic crystals.

7.4.1. We first prove the following preliminary lemma. By the construction of 7.1.2 and proposition 7.3.4 it follows that $f\pi^x$ is integral and monotone for all $x \in [0, 1]$ and all $f \in \mathcal{F}$.

Lemma. *Let $x, y \in [0, 1]$ and let π^x, π^y be the paths defined in section 7.1.2. Then $\mathcal{F}\pi^x \simeq \mathcal{F}\pi^y$.*

Proof. Let J be a finite subset of I . By a direct computation, for all $i \in J$ we obtain:

$$\begin{aligned} d_J(\pi^x, \pi^y) &= \max_{i \in J, t \in [0, 1]} |\alpha_i^\vee(\pi^x(t)) - \alpha_i^\vee(\pi^y(t))| = \\ (22) \quad &= |x - y| \max_{i \in J, t \in [0, 1]} |\alpha_i^\vee((\pi_\lambda \otimes \pi_\mu)(t)) - \alpha_i^\vee(\pi_{\lambda+\mu}(t))| = |x - y| d_J(\pi_\lambda, \pi_\mu, \pi_\nu). \end{aligned}$$

We reduce the distance of x and y so that $d_J(\pi^x, \pi^y) < (1/2c_J)^n$, which by lemma 7.2.2 implies that $\mathcal{F}_J^n \pi^x \simeq \mathcal{F}_J^n \pi^y$. Since n and J are arbitrary the assertion follows. \square

7.4.2. Recall that $\lambda, \mu \in P^+$ and consider $\pi_\lambda \otimes \pi_\mu \in \mathbb{P}_\lambda \otimes \mathbb{P}_\mu$, $\pi_{\lambda+\mu} \in \mathbb{P}_{\lambda+\mu}$. The following obtains by combining lemmata 6.3.5, 7.3.7 and 7.4.1.

Theorem. *The crystals generated by $\pi_\lambda \otimes \pi_\mu$ and $\pi_{\lambda+\mu}$ are isomorphic.*

Remark. The crystals generated by $\pi_\lambda \otimes \pi_\mu$ and $\pi_{\lambda+\mu}$ viewed as paths in \mathbb{P} need not be isomorphic. For example, take $i \in I^{im}$, with $a_{ii} = -1$ and $\lambda \in P^+$, with $\alpha_i^\vee(\lambda) = 1$. Then by definition, $e_i \pi_\lambda = 0$ in \mathbb{P} and so $e_i(\pi_\lambda \otimes \pi_\lambda) = 0$, by the tensor product rules. On the other hand, $e_i \pi_{2\lambda} \neq 0$, again by lemma 5.3.1. Of course $e_i \pi_{2\lambda} = 0$ in $\mathbb{P}_{2\lambda}$.

8. CRYSTAL EMBEDDING THEOREM

We proved that the family of path crystals $\{\mathbb{P}_\lambda \mid \lambda \in P^+\}$ is closed. We will now define the limit \mathbb{P}_∞ of the family $\{\mathbb{P}_\lambda \mid \lambda \in P^+\}$ and show that it is isomorphic to $B(\infty)$ (see theorem 3.3.2).

8.1. The limit \mathbb{P}_∞ .

8.1.1. Let $\lambda, \mu \in P^+$ be two dominant weights and let $\pi \in \mathbb{P}_\lambda$. Denote by $\psi_{\lambda, \lambda+\mu}$ the application $\psi_{\lambda, \lambda+\mu} : \mathbb{P}_\lambda \rightarrow \mathbb{P}_\lambda \otimes \mathbb{P}_\mu$ which sends π to $\pi \otimes \pi_\mu$.

Lemma. *The application $\psi_{\lambda, \lambda+\mu}$ commutes with the $e_i, i \in I$, $\text{wt}(\pi_\lambda \otimes \pi) = \text{wt } \pi + \mu$ and if $f_i \pi \neq 0$, then $f_i \psi_{\lambda, \lambda+\mu}(\pi) = \psi_{\lambda, \lambda+\mu}(f_i \pi)$. Thus $\psi_{\lambda, \lambda+\mu}$ is a crystal embedding up to translation of weight by μ .*

Proof. One has that $\varepsilon_i(\pi_\mu) = 0 \leq \varphi_i(\pi)$, for all $i \in I$. If $\varphi_i(\pi) = 0$, then $f_i \pi = 0$, for all $i \in I$, by section 3.1.4 (3). If $i \in I^{im}$ and $\varphi_i(\pi) = 0$, then $e_i \pi = 0$ by lemma 3.1.5. Then apply the tensor product rules of sections 3.2.1 and 3.2.2. \square

8.1.2. As a set \mathbb{P}_∞ is the inductive limit of the \mathbb{P}_λ with respect to the above embeddings. We will endow \mathbb{P}_∞ with a crystal structure. Let $\pi \in \mathbb{P}_\infty$, then $\pi \in \mathbb{P}_\lambda$ for some $\lambda \in P^+$. Define $e_i\pi$ in \mathbb{P}_∞ as $e_i\pi$ in \mathbb{P}_λ . Define $f_i\pi$ again as in \mathbb{P}_λ but we put $f_i\pi = 0$ only if $f_i\psi_{\lambda, \lambda+\mu}(\pi) = 0$ for all $\mu \in P^+$. Finally, we define the weight of π to be $-\mu$ if $\pi \in (\mathbb{P}_\lambda)_{\lambda-\mu}$. This is clearly well defined. There is a unique element of weight zero, since π_λ is also unique in $(\mathbb{P}_\lambda)_\lambda$. We will denote this element by π_∞ . It satisfies $e_i\pi_\infty = 0$ for all $i \in I$. Notice that we may now forget about the path crystal and consider any closed family of highest weight crystals $\{B(\lambda) | \lambda \in P^+\}$.

8.2. The embedding theorem.

8.2.1. Recall the elementary crystals B_i , $i \in I$ defined in section 3.3.1.

Theorem. *For all $i \in I$ there exists a unique strict embedding $\Psi_i : \mathbb{P}_\infty \longrightarrow \mathbb{P}_\infty \otimes B_i$, sending π_∞ to $\pi_\infty \otimes b_i(0)$.*

Proof. Fix $i \in I$ and $f \in \mathcal{F}$. Call f' a submonomial of f , if f' obtains from f by erasing some of its factors. We say f' is an i -submonomial of f if it obtains by erasing some of the factors f_i in f . Let $\lambda \in P^+$ be such that $\alpha_i^\vee(\lambda) = 0$ and $\alpha_j^\vee(\text{wt } f'\pi_\lambda) > 0$ for all $j \in I \setminus \{i\}$ and for all submonomials f' of f . Let $\mu \in P^+$ be such that $\alpha_j^\vee(\mu) = 0$, for all $j \in I \setminus \{i\}$. We will show that there exists an integer $m \geq 0$ and an i -submonomial f'' of f such that $f(\pi_\lambda \otimes \pi_\mu) = f''\pi_\lambda \otimes f_i^m\pi_\mu$. We argue by induction on the length of f .

For $f = \text{id}$ the assertion is obvious. Let it be true for f and set $f(\pi_\lambda \otimes \pi_\mu) = f''\pi_\lambda \otimes f_i^m\pi_\mu$. First notice that

$$f_i(f''\pi_\lambda \otimes f_i^m\pi_\mu) = \begin{cases} f_i f''\pi_\lambda \otimes f_i^m\pi_\mu, & \varphi_i(f''\pi_\lambda) > \varepsilon_i(f_i^m\pi_\mu), \\ f''\pi_\lambda \otimes f_i^{m+1}\pi_\mu, & \varphi_i(f''\pi_\lambda) > \varepsilon_i(f_i^m\pi_\mu), \end{cases}$$

which is of the required form.

Now for $j \in I \setminus \{i\}$ by assumption we have that $\varphi_j(f''\pi_\lambda) \geq \alpha_j^\vee(\text{wt } f''\pi_\lambda) > 0$. On the other hand, $\varepsilon_j(f_i^m\pi_\mu) = 0$. Indeed, for $j \in I^{im}$ this follows by definition. For $j \in I^{re}$, since $\mu - m\alpha_i + \alpha_j \notin \nu - Q^+$, one has that $e_j f_i^m\pi_\mu = 0$, hence by normality $\varepsilon_j(f_i^m\pi_\mu) = 0$. Then $f_j(f''\pi_\lambda \otimes f_i^m\pi_\mu) = f_j f''\pi_\lambda \otimes f_i^m\pi_\mu$ which is also of the required form.

By section 8.1.2 :

$$\varphi_i(f''\pi_\lambda) = \varepsilon_i(f''\pi_\lambda) + \alpha_i^\vee(\text{wt } f''\pi_\lambda) = \varepsilon_i(f''\pi_\infty) + \alpha_i^\vee(\text{wt } f''\pi_\infty) + \alpha_i^\vee(\lambda) = \varphi_i(f''\pi_\infty),$$

which means that $\varphi_i(f''\pi_\lambda)$ is independent of λ .

Finally one has $\varepsilon_i(b_i(-m)) = \varepsilon_i(f_i^m\pi_\mu)$ (and equal to m for real indices and 0 for imaginary ones) and $\varepsilon_j(b_i(-m)) = -\infty < \varphi_j(f''\pi_\lambda)$, so that if $f(\pi_\lambda \otimes \pi_\mu) = f''\pi_\lambda \otimes f_i^m\pi_\mu$ then also $f(\pi_\infty \otimes b_i(0)) = f''\pi_\infty \otimes b_i(-m)$. \square

8.2.2. **Corollary.** *The crystal \mathbb{P}_∞ is isomorphic as a crystal to $B(\infty)$.*

Proof. Notice that \mathbb{P}_∞ has properties (1)-(4) of definition 3.3.2. Indeed the first three follow by construction and (4) follows by theorem 8.2.1. Then the assertion follows by the uniqueness of $B(\infty)$. \square

9. THE CHARACTER FORMULA

9.1. Weyl-Kac-Borcherds character formula.

9.1.1. Assume the Borcherds-Cartan matrix A to be symmetrizable. Recall $\rho \in \mathfrak{h}^*$ of section 2.1.7. Let $\mathcal{P}(\Pi_{im})$ denote the set of all finite subsets F of Π_{im} such that $\alpha_i^\vee(\alpha_j) = 0$ for all $\alpha_i, \alpha_j \in F$. For all $\lambda \in P^+$ set

$$\mathcal{P}(\Pi_{im})^\lambda = \{F \in \mathcal{P}(\Pi_{im}) \mid \alpha_i^\vee(\lambda) = 0, \text{ for all } \alpha_i \in F\}.$$

Given $F \in \mathcal{P}(\Pi_{im})$, let $|F|$ denote its cardinality and $s(F)$ the sum of its elements. Then the character of the unique irreducible integrable highest weight module of \mathfrak{g} of highest weight $\lambda \in P^+$ is given by the following formula known as the *Weyl-Kac-Borcherds character formula* :

$$(23) \quad \text{char } V(\lambda) = \frac{\sum_{w \in W} \sum_{F \in \mathcal{P}(\Pi_{im})^\lambda} (-1)^{\ell(w)+|F|} e^{w(\lambda+\rho-s(F))}}{\sum_{w \in W} \sum_{F \in \mathcal{P}(\Pi_{im})} (-1)^{\ell(w)+|F|} e^{w(\rho-s(F))}}.$$

9.1.2. **Remark.** It is not known if the above holds when A fails to be symmetrizable. For $\Pi = \Pi_{re}$ of finite cardinality, Kumar [13] and Mathieu [16] independently showed that the right hand side is the correct character formula for the largest integrable quotient of the Verma module of highest weight λ .

9.1.3. Drop the assumption that the Borcherds-Cartan matrix A is symmetrizable. Notice that the right hand side of (23) is still defined in this case. Define the character of \mathbb{P}_λ by

$$\text{char } \mathbb{P}_\lambda = \sum_{\pi \in \mathbb{P}_\lambda} e^{\pi(1)}.$$

Our main result is the following :

Theorem. *The character of \mathbb{P}_λ is given by the Weyl-Kac-Borcherds formula, that is to say the right hand side of (23).*

The rest of the section is devoted to the proof of this theorem.

9.2. The action of the Weyl group. For all $i \in I^{re}$ define \tilde{r}_i on $\pi \in \mathbb{P}_\lambda$ as follows :

$$\tilde{r}_i \pi = \begin{cases} f_i^{\alpha_i^\vee(\pi(1))} \pi, & \text{if } \alpha_i^\vee(\pi(1)) \geq 0, \\ e_i^{-\alpha_i^\vee(\pi(1))} \pi, & \text{if } \alpha_i^\vee(\pi(1)) \leq 0. \end{cases}$$

Then by [15, Section 8] one has that $r_i \mapsto \tilde{r}_i$ extends to a representation $W \rightarrow \text{End}_{\mathbb{Z}} \mathbb{P}_\lambda$ and $w(\pi(1)) = (w\pi)(1)$. Here we note that $\mathbb{P} = \Pi_{\text{int}}$ in the sense of [15] and the root operators $e_i, f_i, i \in I^{re}$ are defined as in [15].

9.3. The Kashiwara function. Recall the crystal $B_J(\infty)$ of section 3.3.3 and that any element in $B_J(\infty)$ takes the form

$$(24) \quad b = \cdots \otimes b_{i_2}(-m_2) \otimes b_{i_1}(-m_1),$$

with $m_k \in \mathbb{N}$ and $m_k = 0$ for $k \gg 0$.

9.3.1. Define the Kashiwara functions on $B_J(\infty)$ through

$$(25) \quad r_i^k(b) = \varepsilon_i(b_{i_k}(-m_k)) - \sum_{j>k} \alpha_i^\vee(\text{wt } b_{i_j}(-m_j)) = \varepsilon_i(b_{i_k}(-m_k)) + \sum_{j>k} m_j a_{i, i_j},$$

noting that this sum is finite since $m_j = 0$ for $j \gg 0$. Observe that $r_i^k(b) \in \{0, -\infty\}$ for $k \gg 0$. Set $R_i(b) = \max_k \{r_i^k(b)\}$. From the definition of J it follows that $R_i(b) \geq 0$ for all $i \in I$ and all $b \in B_J(\infty)$, and $R_i(b) = 0$ for all $i \in I^{im}$ and all $b \in B_J(\infty)$. Note that if $R_i(b) = r_i^{k_0}(b)$ for some k_0 , then $i_{k_0} = i$.

9.3.2. The Kashiwara function determines at which place e_i (resp. f_i) enters when computing $e_i b$ (resp. $f_i b$). Let $\ell_i(b)$ (resp. $s_i(b)$) be the largest (resp. smallest) value of k such that $r_i^k(b) = R_i(b)$. Exactly as in the Kac-Moody case one has the following lemma :

Lemma. For all $b \in B_J(\infty)$ one has :

- (1) $\varepsilon_i(b) = 0$ if and only if $R_i(b) = 0$ for all $i \in I^{re}$ and $\varepsilon_i(b) = 0$ for $i \in I^{im}$.
- (2) For all $i \in I$, f_i enters at the $s_i(b)$ th place.
- (3) For all $i \in I^{re}$, e_i enters at the $\ell_i(b)$ th place.
- (4) For all $i \in I^{im}$, e_i enters at the $s_i(b)$ th place.

Remark. It can happen that $\ell_i(b) = -\infty$ for $i \in I^{re}$, but then simply $e_i b = 0$.

9.3.3. **Lemma.** Let $b, b' \in B_J(\infty)$ be such that $f_i b = f_j b'$ for $i, j \in I^{im}$ and $i \neq j$. Then f_i, f_j commute and there exists $b'' \in B_J(\infty)$ such that $b = f_j b''$.

Proof. Write b, b' as in (24) with m_k replaced by m'_k for the latter. Suppose that f_i enters b at the ℓ th place and f_j enters b' at the ℓ' th place. Since $i \neq j$ we have that $\ell \neq \ell'$. We can assume that $\ell' < \ell$ interchanging i, j if necessary. Note that $f_i b = f_j b'$ forces $m'_\ell = m_\ell + 1 > 0$.

We have $r_j^{\ell'}(b') = \varepsilon_j(b_{i_{\ell'}}(-m'_{\ell'})) + \sum_{s>\ell'} m'_s \alpha_i^\vee(\alpha_{i_s}) = R_j(b') \geq 0$. Since $j = i_{\ell'}$ one has that $\varepsilon_j(b_{i_{\ell'}}(-m'_{\ell'})) = 0$. On the other hand $m'_s \geq 0$ and $\alpha_i^\vee(\alpha_{i_s}) \leq 0$ for all $s > \ell'$, forcing

$m'_s \alpha_i^\vee(\alpha_{i_s}) = 0$ for all $s > \ell'$. In particular, since $i_\ell = i$, $\ell > \ell'$, $m'_\ell > 0$ we obtain that $\alpha_j^\vee(\alpha_i) = 0$, that is $a_{ji} = 0$ and so $a_{ij} = 0$.

Take $c \in B_J(\infty)$. Since $\alpha_j^\vee(\alpha_i) = 0$, one has $r_j^k(c) = r_j^k(f_i c)$, for all $c \in B_J(\infty)$, and all $k \in \mathbb{N}^+$. Then $s_i(f_j c) = s_i(c)$. Similarly $s_j(f_i c) = s_j(c)$ and so $f_i f_j c = f_j f_i c$, as required.

On the other hand, since $f_i b = f_j b'$ we have that $m_{\ell'} = m'_{\ell'} + 1 > 0$. By lemma 9.3.2, e_j enters b at the ℓ' th place and $e_j b \neq 0$. Set $e_j b = b''$, then $b = f_j b''$. Yet $f_j b' = f_i b = f_i f_j b'' = f_j f_i b''$ and so $b' = f_i b''$. \square

9.3.4. Corollary. *Let π, π' be two paths in \mathbb{P}_λ for $\lambda \in P^+$ such that $f_i \pi = f_j \pi'$ for $i, j \in I^{im}$ and $i \neq j$. Then f_i, f_j commute and there exists $\pi'' \in \mathbb{P}_\lambda$ such that $\pi = f_j \pi''$.*

Proof. Embed \mathbb{P}_λ in $B_J(\infty)$:

$$\mathbb{P}_\lambda \xrightarrow{\psi_1} B(\infty) \xrightarrow{\psi_2} B_J(\infty).$$

Then if we assume $f_i \pi = f_j \pi' \neq 0$ in \mathbb{P}_λ f_i, f_j commute with ψ_1 and ψ_2 so that $f_i \psi_2 \psi_1(\pi) = f_j \psi_2 \psi_1(\pi')$. The assertions follow by lemma 9.3.3. We note here that ψ_1 does not in general commute with f_i, f_j that is why we have to assume $f_i \pi = f_j \pi' \neq 0$ (see lemma 8.1.1). \square

9.4. Proof of theorem 9.1.3.

9.4.1. Recall section 9.1; we will show that the character of \mathbb{P}_λ is given by the Weyl-Kac-Borcherds formula. We need to show that :

$$(26) \quad \sum_{\pi \in \mathbb{P}_\lambda} \sum_{w \in W} \sum_{F \in \mathcal{P}(\Pi_{im})} (-1)^{\ell(w) + |F|} e^{w(\rho - s(F)) + \pi(1)} = \sum_{w \in W} \sum_{F \in \mathcal{P}(\Pi_{im})^\lambda} (-1)^{\ell(w) + |F|} e^{w(\lambda + \rho - s(F))}.$$

9.4.2. For all $\mu \in P$ set $O(\mu) = \{(w, F, \pi) \in W \times \mathcal{P}(\Pi_{im}) \times \mathbb{P}_\lambda \mid w(\rho - s(F)) + \pi(1) = \mu\}$. By section 9.2 we have an action of W on $O(\mu)$ by $w(w', F, \pi) = (ww', F, w\pi)$, where $wO(\mu) = O(w\mu)$. Moreover, since $\ell(ww') = \ell(w) + \ell(w') \pmod{2}$, the sum

$$S(\mu) := \sum_{(w, F, \pi) \in O(\mu)} (-1)^{\ell(w) + |F|}$$

satisfies $S(w\mu) = (-1)^{\ell(w)} S(\mu)$. Now the left hand side of (26) becomes

$$(27) \quad \sum_{\mu \in P} S(\mu) e^\mu = \sum_{w \in W} \sum_{\mu \in P^+} (-1)^{\ell(w)} S(\mu) e^{w\mu}.$$

Then we can assume $\mu := w(\rho - s(F)) + \pi(1)$ to be dominant and in this case it remains to show that $S(\mu) = 0$, unless $O(\mu) = \{\text{id}\} \times \mathcal{P}(\Pi_{im}) \times \{\pi_\lambda\}$.

9.4.3. Since μ is dominant and $t \mapsto \pi(t)$ is continuous, either

- (1) there exists some $t \in [0, 1]$ such that $w(\rho - s(F)) + \pi(t)$ is dominant but not regular or
- (2) $w(\rho - s(F)) + \pi(t)$ is regular and dominant for all $t \in [0, 1]$.

Thus define

$O_1(\mu) := \{(w, F, \pi) \in O(\mu) \mid w(\rho - s(F)) + \pi(t) \text{ is dominant but not regular for some } t \in [0, 1]\}$, and

$O_2(\mu) := \{(w, F, \pi) \in O(\mu) \mid w(\rho - s(F)) + \pi(t) \text{ is dominant and regular for all } t \in [0, 1]\}$.

9.4.4. In case (1) exactly as in [15, Theorem 9.1] we obtain that

$$\sum_{(w, F, \pi) \in O_1(\mu)} (-1)^{\ell(w) + |F|} e^\mu = 0.$$

In case (2), $w(\rho - s(F)) + \pi(t)$ being dominant at $t = 0$, implies $w = \text{id}$. Thus we define

$$\tilde{O}_2(\mu) := \{(F, \pi) \in \mathcal{P}(\Pi_{im}) \times \mathbb{P}_\lambda \mid (\text{id}, F, \pi) \in O_2(\mu)\}.$$

The formula we have to prove becomes :

$$(28) \quad \sum_{\mu \in P^+} \sum_{(F, \pi) \in \tilde{O}_2(\mu)} (-1)^{|F|} e^{\rho - s(F) + \pi(1)} = \sum_{F \subset \mathcal{P}(\Pi_{im})^\lambda} (-1)^{|F|} e^{\rho - s(F) + \lambda}.$$

9.4.5. For all $(F, \pi) \in \mathcal{P}(\Pi_{im}) \times \mathbb{P}_\lambda$ set

$$(29) \quad S(F, \pi) := \{\alpha_i \in \Pi_{im} \setminus F \mid \alpha_i^\vee(s(F)) = 0 \text{ and } e_i \pi \neq 0\}.$$

Take $i, j \in I^{im}$ distinct. Notice that if $\alpha_i, \alpha_j \in S(F, \pi)$, then $a_{ij} = a_{ji} = 0$. In particular, $F \cup S(F, \pi) \in \mathcal{P}(\Pi_{im})$. Indeed, since $e_i \pi, e_j \pi \neq 0$, one has $\pi = f_i \pi_1 = f_j \pi_2$, for $\pi_1, \pi_2 \in \mathbb{P}_\lambda$, and the assertion follows by lemma 9.3.4. We call a pair $(F, \pi) \in \mathcal{P}(\Pi_{im}) \times \mathbb{P}_\lambda$ minimal, if $S(F, \pi) = \emptyset$.

For any subset $S = \{\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k}\} \subset \mathcal{P}(\Pi_{im})$, set $f_S := f_{i_1} f_{i_2} \dots f_{i_k}$ and similarly $e_S := e_{i_1} e_{i_2} \dots e_{i_k}$. Notice that since the f_{i_j} (resp. e_{i_j}) mutually commute, the monomial f_S (resp. e_S) does not depend on the order of the indices. Suppose that $\pi \in \mathbb{P}_\lambda$ satisfies $e_i \pi \neq 0$, for all $\alpha_i \in S$. Then $e_S \pi \neq 0$. Indeed this follows from lemmata 9.3.2 and 8.1.1. Again if $f_i \pi \neq 0$ for all $\alpha_i \in S$, then $f_S \pi \neq 0$. This follows from lemma 2.1.11 (2) and section 6.3.3.

9.4.6. For all $(F, \pi) \in \mathcal{P}(\Pi_{im}) \times \mathbb{P}_\lambda$, set $(F_0, \pi_0) = (F \cup S(F, \pi), e_{S(F, \pi)} \pi)$. Clearly (F_0, π_0) is minimal. For a minimal element (F_0, π_0) define $F_0^{\pi_0} := \{S \subset F_0 \mid \forall \alpha_i \in S, f_i \pi \neq 0\}$. Then set

$$\Omega(F_0, \pi_0) = \{(F_0 \setminus S, f_S \pi_0) \mid S \in F_0^{\pi_0}\}.$$

The following is straightforward.

Lemma. *The only minimal element in $\Omega(F_0, \pi_0)$ is (F_0, π_0) . Moreover, if $(F, \pi) = (F_0 \setminus S, f_S \pi_0) \in \Omega(F_0, \pi_0)$, for $S \in F_0^{\pi_0}$, then $S(F, \pi) = S$.*

An immediate consequence of the above is that for any two minimal elements $(F_0, \pi_0) \neq (F'_0, \pi'_0)$, one has $\Omega(F_0, \pi_0) \cap \Omega(F'_0, \pi'_0) = \emptyset$.

Remark. Note that for all $(F, \pi) \in \Omega(F_0, \pi_0)$, the weight $-s(F) + \pi(1)$ is fixed (but it does not uniquely define $\Omega(F_0, \pi_0)$!).

9.4.7. We show in section 9.4.10 that if $\Omega(F_0, \pi_0) \cap \tilde{O}_2(\mu) \neq \emptyset$, then $\Omega(F_0, \pi_0) \subset \tilde{O}_2(\mu)$. Then if we set $\Omega(\mu) = \{(F_0, \pi_0) \mid \Omega(F_0, \pi_0) \subset \tilde{O}_2(\mu)\}$ we have

$$(30) \quad \tilde{O}_2(\mu) = \coprod_{(F_0, \pi_0) \in \Omega(\mu)} \Omega(F_0, \pi_0).$$

Admit 9.4.10, so then (30) holds. We have :

$$(31) \quad \sum_{(F, \pi) \in \tilde{O}_2(\mu)} (-1)^{|F|} = \sum_{(F_0, \pi_0) \in \Omega(\mu)} \left(\sum_{(F, \pi) \in \Omega(F_0, \pi_0)} (-1)^{|F|} \right).$$

We will compute the following sum :

$$(32) \quad \Sigma := \sum_{(F, \pi) \in \Omega(F_0, \pi_0)} (-1)^{|F|} e^{\rho - s(F) + \pi(1)}.$$

9.4.8. **Lemma.** *If $|\Omega(F_0, \pi_0)| > 1$, then the sum Σ above is zero.*

Proof. Write F_0 as $F_0 = F'_0 \sqcup F''_0$, where $F'_0 := \{\alpha_i \in F_0 \mid \alpha_i^\vee(\pi(1)) \neq 0\}$ (equivalently, by 5.3.1, $F'_0 := \{\alpha_i \in F_0 \mid f_i \pi \neq 0\}$). Our hypothesis that $|\Omega(F_0, \pi_0)| > 1$ implies that $F'_0 \neq \emptyset$. Set $|F'_0| = n \geq 1$. Then the cardinality of $\Omega(F_0, \pi_0)$ is equal to the number of subsets of F'_0 . Moreover, the coefficient of $e^{\rho - s(F) + \pi(1)}$ in Σ is

$$(-1)^{|F''_0|} ((-1)^n + (-1)^{n-1} \binom{n}{1} + (-1)^{n-2} \binom{n}{2} + \cdots + (-1) \binom{n}{n-1} + 1) = 0.$$

□

9.4.9. **Lemma.** *If $|\Omega(F_0, \pi_0)| = 1$, then $\pi_0 = \pi_\lambda$ and $F_0 \in \mathcal{P}(\Pi_{im})^\lambda$.*

Proof. Let $\Omega(F_0, \pi_0) = \{(F_0, \pi_0)\}$ be a singleton and recall section 6.3.3. Then for all $\alpha_j \in F_0$, one has that $\alpha_j^\vee(\pi_0(1)) = 0$. In particular, if $\pi_0 = f_i \pi$ for some $\pi \in \mathbb{P}_\lambda$ and $i \in I$, then $\alpha_i^\vee(s(F_0)) = 0$ and $\alpha_j^\vee(\pi(1)) = 0$, for all $\alpha_j \in F_0$.

Assume that $\pi_0 \neq \pi_\lambda$. Then we may write $\pi_0 = f_i \pi$ as above. Suppose that $i \in I^{im}$, then $e_i \pi_0 \neq 0$ and by the above remark $\alpha_i^\vee(s(F_0)) = 0$. By the minimality of (F_0, π_0) , this implies that $\alpha_i \in F_0$. Yet $\pi_0 = f_i \pi$ implies $f_i \pi_0 \neq 0$ by lemma 4.1.6 and so $(F_0 \setminus \{i\}, f_i \pi_0) \in \Omega(F_0, \pi_0)$, which contradicts the hypothesis.

Let now $i \in I^{re}$. Then $\alpha_i^\vee(\rho - s(F_0) + \pi_0(t)) = 1 + \alpha_i^\vee(\pi_0(t)) = 1 + h_i^{\pi_0}(t)$. But $e_i\pi_0 \neq 0$ which means (by definition) that $h_i^{\pi_0}(t)$ takes integral values ≤ -1 . Hence $\rho - s(F_0) + \pi_0(t)$ is not regular for all $t \in [0, 1]$, again a contradiction.

We obtain that $\pi_0 = \pi_\lambda$ and $F_0 \in \mathcal{P}(\Pi_{im})^\lambda$. \square

By lemmata 9.4.8, 9.4.9 the only remaining terms in the left hand side of (28) is the right hand side. Thus to complete the proof of theorem 9.1.3 it remains to prove (30). As we noted in section 9.4.7, this follows by the lemma below.

9.4.10. **Lemma.** *Let $\Omega(F_0, \pi_0) \cap \tilde{O}_2(\mu) \neq \emptyset$. Then $\Omega(F_0, \pi_0) \subset \tilde{O}_2(\mu)$.*

Proof. Fix $(F, \pi) \in \Omega(F_0, \pi_0) \cap \tilde{O}_2(\mu)$.

Assume that $(F \cup \{\alpha_i\}, e_i\pi) \in \Omega(F_0, \pi_0)$. This means that $\alpha_i \notin F$, $\alpha_i^\vee(s(F)) = 0$ and $e_i\pi \neq 0$. We show that $(F \cup \{\alpha_i\}, e_i\pi) \in \tilde{O}_2(\mu)$.

By definition of $e_i\pi$, there exists a piecewise linear function $c(t)$ with $0 \leq c(t) \leq 1$ for all $t \in [0, 1]$ such that $e_i\pi(t) = \pi(t) + c(t)\alpha_i$. Then since $\rho - s(F) + \pi(t)$ is regular and dominant for all $t \in [0, 1]$ and α_i is anti-dominant, we obtain that $\rho - s(F \cup \{\alpha_i\}) + e_i\pi(1) = \rho - s(F) - \alpha_i + \pi(t) + c(t)\alpha_i = \rho - s(F) + \pi(t) + (c(t) - 1)\alpha_i$ is also regular and dominant for all $t \in [0, 1]$, as required.

Now suppose that $(F \setminus \{\alpha_i\}, f_i\pi) \in \Omega(F_0, \pi_0)$. It follows that $\alpha_i \in F$ and $f_i\pi \neq 0$. We show that $(F \setminus \{\alpha_i\}, f_i\pi) \in \tilde{O}_2(\mu)$.

Set $F' = F \setminus \{\alpha_i\}$, then $F = F' \cup \{\alpha_i\}$. By assumption, $M(t) := \rho - s(F) + \pi(t) = \rho - s(F') - \alpha_i + \pi(t)$ is regular and dominant for all $t \in [0, 1]$. We need to show that $M'(t) := \rho - s(F) + \alpha_i + (f_i\pi)(t) = \rho - s(F') + (f_i\pi)(t)$ is regular and dominant for all $t \in [0, 1]$. Now for $t \in [f_-^i(\pi), 1]$ one has $f_i\pi(t) = \pi(t) - \alpha_i$ and so $M'(t) = M(t)$, hence $M'(t)$ is regular and dominant for all $t \in [f_-^i(\pi), 1]$.

Suppose that for some $t \in [0, f_-^i(\pi)]$, $M'(t) = \rho - s(F') + (f_i\pi)(t)$ is not regular. This means that there exists $j \in I^{re}$ such that $\alpha_j^\vee(M'(t)) = 0$, for some $t \in [0, f_-^i(\pi)[$. In this region, $(f_i\pi)(t) = r_i\pi(t)$, hence

$$(33) \quad h_j(t) := \alpha_j^\vee(M'(t)) = h_j^{f_i\pi}(t) + \alpha_j^\vee(\rho - s(F')) = \alpha_j^\vee(\pi(t)) - \alpha_i^\vee(\pi(t))a_{ji} + \alpha_j^\vee(\rho - s(F')) = 0,$$

for some $t \in [0, f_-^i(\pi)]$. On the other hand $h_j(0) = \alpha_j^\vee(\rho - s(F')) > 0$ and $h_j(f_-^i(\pi)) = \alpha_j^\vee(M'(f_-^i(\pi))) = \alpha_j^\vee(M(f_-^i(\pi))) > 0$, hence the function h_j attains a local minimum at some $t_0 \in]0, f_-^i(\pi)[$ and consequently $h_j^{f_i\pi}$ attains a local minimum at t_0 .

Let $\pi = (\lambda_1, \lambda_2, \dots, \lambda_s; 0, a_1, a_2, \dots, a_s = 1)$ and recall proposition 6.1.3, choosing p as defined there. One has $a_{p-1} < f_-^i(\pi) \leq a_p$ and

$$f_i\pi = (r_i\lambda_1, r_i\lambda_2, \dots, r_i\lambda_p, \lambda_p, \dots, \lambda_s; 0, a_1, \dots, a_{p-1}, f_-^i(\pi), a_p, \dots, a_s = 1).$$

By lemma 5.3.7, we must have $t_0 = a_k$ for some $k \leq p-1$ and so either $\alpha_j^\vee(\lambda_k) \leq 0$ and $\alpha_j^\vee(\lambda_{k+1}) > 0$, or $\alpha_j^\vee(\lambda_k) < 0$ and $\alpha_j^\vee(\lambda_{k+1}) \geq 0$, depending on whether the minimum at t_0 is right or left. Then by lemma 5.3.2, if $\lambda_{k+1} \xleftarrow{\beta_t} \dots \xleftarrow{\beta_1} \lambda_k$, we obtain $\beta_\ell = \alpha_j$ for some ℓ , with $1 \leq \ell \leq t$. On the other hand, by proposition 6.1.3, $\alpha_i^\vee(\lambda_k) = \alpha_i^\vee(\lambda_{k+1})$, for all k , with

$1 \leq k \leq p-1$ and so $\alpha_i^\vee(\beta_s) = 0$ for all s , with $1 \leq s \leq t$. In particular $a_{ij} = 0$ and so $h_j(t) = \alpha_j^\vee(M(t))$ which is strictly positive by assumption. This contradiction proves that $M'(t)$ is regular for all $t \in [0, 1]$. \square

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